

DEFORMATION TECHNIQUES FOR SPARSE SYSTEMS

GABRIELA JERONIMO^{1,2}, GUILLERMO MATERA^{2,3}, PABLO SOLERNÓ^{1,2}, AND ARIEL
WAISSBEIN^{4,5}

ABSTRACT. We exhibit a probabilistic symbolic algorithm for solving zero-dimensional sparse systems. Our algorithm combines a symbolic homotopy procedure, based on a flat deformation of a certain morphism of affine varieties, with the polyhedral deformation of Huber and Sturmfels. The complexity of our algorithm is quadratic in the size of the combinatorial structure of the input system. This size is mainly represented by the mixed volume of Newton polytopes of the input polynomials and an arithmetic analogue of the mixed volume associated to the deformations under consideration.

1. INTRODUCTION

Numeric and symbolic methods for computing all solutions of a given zero-dimensional polynomial system usually rely on deformation techniques, based on a perturbation of the original system and a subsequent (numeric or symbolic) path-following method (see, e.g., [1], [3], [6], [24], [32], [35]). The complexity of such algorithms is usually determined by geometric invariants associated to the family of systems under consideration (see, e.g., [16], [25], [53], [44], [24], [27], [13], [50], [31], [39]), typically in the form of a suitable (arithmetic or geometric) Bézout number (see [36], [25], [33], [46], [26], [23], [43]).

Sparse elimination theory is concerned with finding bounds for such Bézout numbers in the case of a sparse polynomial system. Its origins can be traced back to the results by D.N. Bernstein, A.G. Kushnirenko and A.G. Khovanski ([4], [29], [28]) that bound the number of solutions of a polynomial system in terms of certain combinatorial invariants. More precisely, the Bernstein–Kushnirenko–Khovanski (BKK for short) theorem asserts that the number of isolated solutions in the n -dimensional complex torus $(\mathbb{C}^*)^n$ of a polynomial system of n equations in n unknowns is bounded by the mixed volume of the family of Newton polytopes of the corresponding polynomials.

Numeric (homotopy continuation) methods for sparse systems are typically based on a family of deformations called polyhedral homotopies ([25], [54], [53]). Polyhedral homotopies preserve the Newton polytope of the input polynomials and

Date: February 2, 2008.

1991 *Mathematics Subject Classification.* Primary 14Q05, 52B20, 68W30; Secondary 12Y05, 13F25, 14Q20, 68W40.

Key words and phrases. Sparse system solving, symbolic homotopy algorithms, polyhedral deformations, mixed volume, nonarchimedean height, Puiseux expansions of space curves, Newton–Hensel lifting, geometric solutions, probabilistic algorithms, complexity.

Research was partially supported by the following grants: UBACyT X112 (2004–2007), UBACyT X847 (2006–2009), PIP CONICET 2461, PIP CONICET 5852/05, UNGS 30/3005 and MTM2004-01167 (2004–2007).

yield an effective version of the BKK theorem (see e.g. [25], [26]). More precisely, suppose that we are given a zero-dimensional $(\Delta_1, \dots, \Delta_n)$ -sparse system defined by n polynomials f_1, \dots, f_n in n variables, where $\Delta_1, \dots, \Delta_n$ are the supports of f_1, \dots, f_n , and let $V \subset (\mathbb{C}^*)^n$ be the variety defined by the common zeros of f_1, \dots, f_n over $(\mathbb{C}^*)^n$. Then a polyhedral homotopy consists in an algebraic curve $W \subset (\mathbb{C}^*)^{n+1}$ such that the projection $\pi : W \rightarrow \mathbb{C}^*$ onto the first coordinate is dominant with generically finite fibers whose degree is the mixed volume $MV(\text{conv}(\Delta_1), \dots, \text{conv}(\Delta_n))$ of the convex hulls of $\Delta_1, \dots, \Delta_n$, the identity $\pi^{-1}(1) = \{1\} \times V$ holds and the first terms of the Puiseux expansions of the branches of W lying above 0 can be easily computed. Numerical continuation methods compute the first terms of these Puiseux expansions and then follow the branches of W along the interval $[0, 1]$ to obtain approximations to all the points of the input variety V .

From the symbolic point of view, a family of homotopy algorithms is based on a flat deformation of a certain morphism of affine varieties. This deformation, implicitly considered in the papers [20], [19], is isolated in [24] and refined in [50], [23], [7], [39], in order to solve particular instances of a parametric system with a finite generically-unramified linear projection of low degree. More precisely, let V be a zero-dimensional variety of \mathbb{C}^n , defined by a “square” system $f_1 = \dots = f_n = 0$, and let be given an algebraic curve $W \subset \mathbb{C}^{n+1}$ and a (dominant) projection mapping $\pi : W \rightarrow \mathbb{C}$ which represent a deformation of V . Then, from a complete description of a generic fiber of the projection $\pi : W \rightarrow \mathbb{C}$, it is possible to compute a complete description of the input fiber, say $\pi^{-1}(1) = \{1\} \times V$. The complexity of this procedure can be roughly estimated by the product of two geometric invariants: the degree of the morphism π and the degree of the curve W . The algorithm is nearly optimal in worst case [13], and has good performance over certain well-posed families of polynomial systems of practical interest (see [24], [50], [7], [12]).

In this article we combine these symbolic techniques, particularly in the version of [7], with the polyhedral deformation [25], in order to derive a symbolic probabilistic algorithm for solving sparse zero-dimensional polynomial systems with quadratic complexity in the size of the combinatorial structure of the input system. More precisely, suppose that we are given polynomials f_1, \dots, f_n of $\mathbb{Q}[X_1, \dots, X_n]$ such that the system $f_1 = 0, \dots, f_n = 0$ defines a zero-dimensional affine subvariety V of \mathbb{C}^n . Denote by $\Delta_1, \dots, \Delta_n \subset \mathbb{Z}_{\geq 0}^n$ the supports of f_1, \dots, f_n , and assume that $0 \in \Delta_i$ for $1 \leq i \leq n$ and the mixed volume D of the Newton polytopes $\text{conv}(\Delta_1), \dots, \text{conv}(\Delta_n)$ is nonzero. Then, given a “sufficiently generic” lifting function ω for $(\Delta_1, \dots, \Delta_n)$ and the corresponding mixed subdivision, we exhibit an algorithm which computes a complete description of the solution set V of the input system $f_1 = 0, \dots, f_n = 0$.

The polyhedral deformation under consideration requires that the coefficients of the input polynomials satisfy certain generic conditions (see Section 3.1). For this reason, we introduce some auxiliary *generic* polynomials g_1, \dots, g_n with the same supports $\Delta_1, \dots, \Delta_n$ and consider the perturbed polynomial system $h_i := f_i + g_i$ for $1 \leq i \leq n$. We first solve the system $h_1 = 0, \dots, h_n = 0$ and then recover the solutions to the input system $f_1 = 0, \dots, f_n = 0$ by considering a homotopy of type $f_1 + (1 - T)g_1, \dots, f_n + (1 - T)g_n$ (in Section 5).

The system $h_1 = 0, \dots, h_n = 0$ is solved by considering the polyhedral homotopy of [25]. This homotopy introduces a new variable T and deforms the polynomial

h_i by multiplying each nonzero monomial of h_i by the power of T determined by the given lifting function ω . From the genericity of the coefficients of h_1, \dots, h_n we conclude that the roots of the resulting parametric system $\hat{h}_1 = 0, \dots, \hat{h}_n = 0$ are algebraic functions of the parameter T whose expansions as Puiseux series can be obtained by “lifting” the solutions to certain zero-dimensional polynomial systems $h_{1,\gamma}^{(0)} = \dots = h_{n,\gamma}^{(0)} = 0$ associated to the lower facets \hat{C}_γ of the lifted polytopes $\text{conv}(\hat{\Delta}_1), \dots, \text{conv}(\hat{\Delta}_n)$ defined by ω . These polynomial systems $h_{1,\gamma}^{(0)} = \dots = h_{n,\gamma}^{(0)} = 0$ can be easily solved due to their specific structure, which enables us to use their solutions as a starting point for our computations (see Section 4.1 for details).

The complexity of our algorithm is mainly expressed in terms of two quantities related to the combinatorial structure of the input system: the mixed volume $D := M(\text{conv}(\Delta_1), \dots, \text{conv}(\Delta_n))$ and certain (nonarchimedean) heights E, E' associated to our polyhedral deformations. These heights, which are an arithmetic analogue of the mixed volume D (see [41], [42]), can be bounded in terms of certain mixed volumes associated to the polyhedral deformation under consideration, with equality for a generic choice of the coefficients of the polynomials \hat{h}_i (see Lemma 2.3 below; compare also with [43, Theorem 1.1]). Therefore, we may paraphrase our complexity estimate as saying that it is *quadratic* in the combinatorial structure of the input system, with a geometric and an arithmetic component. More precisely, our algorithm requires $\mathcal{L}n^{O(1)}D \max\{E, E'\}$ arithmetic operations over \mathbb{Q} (up to polylogarithmic terms), where \mathcal{L} is the number of arithmetic operations required to evaluate the polynomials \hat{h}_i and $f_i + g_i$, and E and E' denote the height of the varieties defined by $\hat{h}_1 = 0, \dots, \hat{h}_n = 0$ and $f_1 + (1-T)g_1 = 0, \dots, f_n + (1-T)g_n = 0$ respectively.

This improves and refines the estimate of [7] in the case of a sparse system, which is expressed as a fourth power of D and the maximum of the degrees of the varieties $\hat{h}_1 = 0, \dots, \hat{h}_n = 0$ and $f_1 + (1-T)g_1 = 0, \dots, f_n + (1-T)g_n = 0$. We observe that this maximum is an upper bound for the heights E and E' respectively. On the other hand, it also improves [44], [45], which solve a sparse system with a complexity which is roughly quartic in the size of the combinatorial structure of the input system. Finally, we provide an explicit estimate of the error probability of all the steps of our algorithm. This might be seen as a further contribution to the symbolic stage of the probabilistic seminumeric method of [25], which lacks such analysis of the error probability.

2. NOTIONS AND NOTATIONS

2.1. Sparse Elimination. Here we introduce some notions and notations of convex geometry and sparse elimination theory (see e.g. [18], [25]) that will be used in the sequel.

Let X_1, \dots, X_n be indeterminates over \mathbb{Q} and write $X := (X_1, \dots, X_n)$. For $q := (q_1, \dots, q_n) \in \mathbb{Z}^n$, we use the notation $X^q := X_1^{q_1} \dots X_n^{q_n}$. Let $f := \sum_q c_q X^q$ be a Laurent polynomial in $\mathbb{Q}[X, X^{-1}] := \mathbb{Q}[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$. By the **support** of f we understand the subset of \mathbb{Z}^n defined by the elements $q \in \mathbb{Z}^n$ for which $c_q \neq 0$ holds. The **Newton polytope** of f is the convex hull of the support of f in \mathbb{R}^n .

A sparse polynomial system with respect to *a priori* fixed finite subsets $\Delta_1, \dots, \Delta_n$ of $(\mathbb{Z}_{\geq 0})^n$ is defined by polynomials

$$f_i(X) := \sum_{q \in \Delta_i} a_{i,q} X^q \quad (1 \leq i \leq n),$$

with $a_{i,q} \in \mathbb{C} \setminus \{0\}$ for each $q \in \Delta_i$ and $1 \leq i \leq n$.

For a finite subset Δ of \mathbb{Z}^n , we denote by $Q := \text{conv}(\Delta)$ its convex hull in \mathbb{R}^n . The usual Euclidean volume of a polytope Q in \mathbb{R}^n will be denoted by $\text{vol}_{\mathbb{R}^n}(Q)$.

Let Q_1, \dots, Q_n be convex polytopes in \mathbb{R}^n . For $\lambda_1, \dots, \lambda_n \in \mathbb{R}_{\geq 0}$, we use the notation $\lambda_1 Q_1 + \dots + \lambda_n Q_n$ to refer to the Minkowski sum $\lambda_1 Q_1 + \dots + \lambda_n Q_n := \{x \in \mathbb{R}^n : \exists x_1 \in Q_1, \dots, \exists x_n \in Q_n \text{ such that } x = \lambda_1 x_1 + \dots + \lambda_n x_n\}$. Consider the real-valued function $(\lambda_1, \dots, \lambda_n) \mapsto \text{vol}_{\mathbb{R}^n}(\lambda_1 Q_1 + \dots + \lambda_n Q_n)$. This is a homogeneous polynomial function of degree n in the λ_i (see e.g. [14, Chapter 7, Proposition §4.4.9]). The mixed volume $MV(Q_1, \dots, Q_n)$ of Q_1, \dots, Q_n is defined as the coefficient of the monomial $\lambda_1 \dots \lambda_n$ in $\text{vol}_{\mathbb{R}^n}(\lambda_1 Q_1 + \dots + \lambda_n Q_n)$.

For $i = 1, \dots, n$, let Δ_i be a finite subset of $\mathbb{Z}_{\geq 0}^n$ and let $Q_i := \text{conv}(\Delta_i)$ denote the corresponding polytope. Let f_1, \dots, f_n be a sparse polynomial system with respect to $\Delta_1, \dots, \Delta_n$. The BKK theorem ([4], [29], [28]) asserts that the system $f_1 = 0, \dots, f_n = 0$ has at most $MV(Q_1, \dots, Q_n)$ isolated common solutions in the n -dimensional torus $(\mathbb{C}^*)^n$, with equality for generic choices of the coefficients of f_1, \dots, f_n . Furthermore, if the condition $0 \in Q_i$ holds for $1 \leq i \leq n$, then $MV(Q_1, \dots, Q_n)$ bounds the number of solutions in the n -dimensional affine complex space $\mathbb{A}^n := \mathbb{A}^n(\mathbb{C})$ (see [33]).

Assume that the union of the sets $\Delta_1, \dots, \Delta_n$ affinely generate \mathbb{Z}^n , and consider the partition of $\Delta_1, \dots, \Delta_n$ defined by the relation $\Delta_i \sim \Delta_j$ if and only if $\Delta_i = \Delta_j$. Let $s \in \mathbb{N}$ denote the number of classes in this partition, and let $\mathcal{A}^{(1)}, \dots, \mathcal{A}^{(s)} \subset \mathbb{Z}^n$ denote a member in each class. Write $\mathcal{A} := (\mathcal{A}^{(1)}, \dots, \mathcal{A}^{(s)})$. For $\ell = 1, \dots, s$, let $k_\ell := \#\{i : \Delta_i = \mathcal{A}^{(\ell)}\}$. Without loss of generality, we will assume that $\Delta_1 = \dots = \Delta_{k_1} = \mathcal{A}^{(1)}$, $\Delta_{k_1+1} = \dots = \Delta_{k_1+k_2} = \mathcal{A}^{(2)}$ and so on.

A cell of \mathcal{A} is a tuple $C = (C^{(1)}, \dots, C^{(s)})$ with $C^{(\ell)} \neq \emptyset$ and $C^{(\ell)} \subset \mathcal{A}^{(\ell)}$ for $1 \leq \ell \leq s$. We define

$$\begin{aligned} \text{type}(C) &:= (\dim(\text{conv}(C^{(1)})), \dots, \dim(\text{conv}(C^{(s)}))), \\ \text{conv}(C) &:= \text{conv}(C^{(1)} + \dots + C^{(s)}), \\ \#(C) &:= \#(C^{(1)}) + \dots + \#(C^{(s)}), \\ \text{vol}_{\mathbb{R}^n}(C) &:= \text{vol}_{\mathbb{R}^n}(\text{conv}(C)). \end{aligned}$$

A face of a cell C is a cell $\mathcal{C} = (\mathcal{C}^{(1)}, \dots, \mathcal{C}^{(s)})$ of C with $\mathcal{C}^{(\ell)} \subset C^{(\ell)}$ for $1 \leq \ell \leq s$ such that there exists a linear functional $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}$ that takes its minimum over $C^{(\ell)}$ at $\mathcal{C}^{(\ell)}$ for $1 \leq \ell \leq s$. One such functional γ is called an **inner normal** of C .

A mixed subdivision of \mathcal{A} is a collection of cells $\mathfrak{C} = \{C_1, \dots, C_m\}$ of \mathcal{A} satisfying conditions (1)–(4) below:

- (1) $\dim(\text{conv}(C_j)) = n$ for $1 \leq j \leq m$,
- (2) the intersection $\text{conv}(C_i) \cap \text{conv}(C_j) \subset \mathbb{R}^n$ is either the empty set or a face of both $\text{conv}(C_i)$ and $\text{conv}(C_j)$ for $1 \leq i < j \leq m$,
- (3) $\bigcup_{j=1}^m \text{conv}(C_j) = \text{conv}(\mathcal{A})$,
- (4) $\sum_{\ell=1}^s \dim(\text{conv}(C_j^{(\ell)})) = n$ for $1 \leq j \leq m$.

If \mathfrak{C} also satisfies the condition

$$(5) \#(C_j) = n + s \text{ for } 1 \leq j \leq m,$$

we say that \mathfrak{C} is a **fine-mixed subdivision** of \mathcal{A} . Observe that, as a consequence of conditions (4) and (5), for each cell $C_j = (C_j^{(1)}, \dots, C_j^{(s)})$ in a fine-mixed subdivision the identity $\dim(\text{conv}(C_j^{(\ell)})) = \#C_j^{(\ell)} - 1$ holds for $1 \leq \ell \leq s$.

We point out that a mixed subdivision \mathfrak{C} of \mathcal{A} enables us to compute the mixed volume of the family $Q_1 = \text{conv}(\Delta_1), \dots, Q_n = \text{conv}(\Delta_n)$ by means of the following identity (see [25, Theorem 2.4.]):

$$(2.1) \quad MV(Q_1, \dots, Q_n) = \sum_{\substack{C_i \in \mathfrak{C} \\ \text{type}(C_i) = (k_1, \dots, k_s)}} k_1! \dots k_s! \cdot \text{vol}_{\mathbb{R}^n}(C_i).$$

A fine-mixed subdivision of \mathcal{A} can be obtained by means of a lifting process as explained in what follows. For $1 \leq \ell \leq s$, let $\omega_\ell : \mathcal{A}^{(\ell)} \rightarrow \mathbb{R}$ be an arbitrary function. The tuple $\omega := (\omega_1, \dots, \omega_s)$ is called a **lifting function** for \mathcal{A} . Once a lifting function ω is fixed, the graph of any subset $C^{(\ell)}$ of $\mathcal{A}^{(\ell)}$ will be denoted by $\widehat{C}^{(\ell)} := \{(q, \omega_\ell(q)) \in \mathbb{R}^{n+1} : q \in C^{(\ell)}\}$. Then, for a sufficiently generic lifting function ω , the set of cells C of \mathcal{A} satisfying the conditions:

- (i) $\dim(\text{conv}(\widehat{C}^{(1)} + \dots + \widehat{C}^{(s)})) = n$,
- (ii) $(\widehat{C}^{(1)}, \dots, \widehat{C}^{(s)})$ is a face of $(\widehat{\mathcal{A}}^{(1)}, \dots, \widehat{\mathcal{A}}^{(s)})$ whose inner normal has positive last coordinate,

is a fine-mixed subdivision of \mathcal{A} (see [25, Section 2]). More precisely, we have the following result (cf. [25, Section 2]):

Lemma 2.1. *The lifting process associated to a lifting function ω yields a fine-mixed subdivision of \mathcal{A} if the following condition holds: for every $r_1, \dots, r_s \in \mathbb{Z}_{\geq 0}$ with $\sum_{\ell=1}^s r_\ell > n$ and every cell $(C^{(1)}, \dots, C^{(s)})$ with $C^{(\ell)} := \{q_{\ell,0}, \dots, q_{\ell,r_\ell}\} \subset \mathcal{A}^{(\ell)}$ ($1 \leq \ell \leq s$), if*

$$V(C) := \begin{pmatrix} q_{1,1} - q_{1,0} \\ \vdots \\ q_{1,r_1} - q_{1,0} \\ \vdots \\ q_{s,1} - q_{s,0} \\ \vdots \\ q_{s,r_s} - q_{s,0} \end{pmatrix} \quad \text{and} \quad V(\widehat{C}) := \begin{pmatrix} q_{1,1} - q_{1,0} & \omega_1(q_{1,1}) - \omega_1(q_{1,0}) \\ \vdots & \vdots \\ q_{1,r_1} - q_{1,0} & \omega_1(q_{1,r_1}) - \omega_1(q_{1,0}) \\ \vdots & \vdots \\ q_{s,1} - q_{s,0} & \omega_s(q_{s,1}) - \omega_s(q_{s,0}) \\ \vdots & \vdots \\ q_{s,r_s} - q_{s,0} & \omega_s(q_{s,r_s}) - \omega_s(q_{s,0}) \end{pmatrix},$$

then

$$\text{rank}(V(C)) = n \implies \text{rank}(V(\widehat{C})) = n + 1.$$

Proof. Notice that (1)–(3) are automatically satisfied by the set of cells defined by conditions (i)–(ii). Assume that the condition of the statement of the lemma is met and consider a cell $C = (C^{(1)}, \dots, C^{(s)})$ of \mathcal{A} satisfying conditions (i) and (ii) above. Being \widehat{C} a lower facet of \mathcal{A} , the identity $\dim(\text{conv}(C^{(1)} + \dots + C^{(s)})) = \dim(\text{conv}(\widehat{C}^{(1)} + \dots + \widehat{C}^{(s)}))$ must hold. Write $C^{(\ell)} = \{q_{\ell,0}, \dots, q_{\ell,r_\ell}\}$ for $1 \leq \ell \leq s$. Then we have that $\text{rank}(V(C)) = \dim(\langle q_{\ell,j} - q_{\ell,0} : 1 \leq \ell \leq r_\ell, 1 \leq j \leq r_j \rangle) = \dim(\text{conv}(C^{(1)} + \dots + C^{(s)})) = n$ and $\text{rank}(V(\widehat{C})) = \dim(\text{conv}(\widehat{C}^{(1)} + \dots + \widehat{C}^{(s)})) = n$. Now, the condition stated on ω implies that $\sum_{\ell=1}^s r_\ell \leq n$ and, taking into account that the $\sum_{\ell=1}^s r_\ell$ many vectors $q_{\ell,j} - q_{\ell,0}$ ($1 \leq \ell \leq s, 1 \leq j \leq r_\ell$) span

a linear space of dimension n , we conclude that the equality $\sum_{\ell=1}^s r_\ell = n$ holds, which shows that condition (5) in the definition of a fine-mixed subdivision is met. Moreover, as $\sum_{\ell=1}^s \dim(\text{conv}(C^{(\ell)})) \geq \dim(\text{conv}(C^{(1)} + \dots + C^{(s)}))$ for an arbitrary cell C , we see that $\dim(\text{conv}(C^{(\ell)})) = r_\ell$ holds for every $1 \leq \ell \leq s$, which implies that condition (4) is also valid. This finishes the proof of the lemma. \square

Note that the condition $\text{rank}(V(\widehat{C})) = n+1$ can be restated as the non-vanishing of the maximal minors of the matrix $V(\widehat{C})$. Since $\text{rank}(V(C)) = n$, these maximal minors are nonzero linear forms in the unknown values $\omega_\ell(q_{\ell,j})$ of the lifting function. Thus, if $\mathcal{N}_\ell = \#\mathcal{A}^{(\ell)}$ for every $1 \leq \ell \leq s$, a sufficiently generic lifting function can be obtained by randomly choosing the values $\omega_\ell(q_{\ell,j})$ of ω at the points of $\mathcal{A}^{(\ell)}$ from the set $\{1, 2, \dots, \rho^{2^{\mathcal{N}_1} + \dots + \mathcal{N}_s}}\}$, with probability of success at least $1 - 1/\rho$ for $\rho \in \mathbb{N}$.

In the sequel, we shall assume that a sufficiently generic lifting function and the induced fine-mixed subdivision of \mathcal{A} are given.

2.2. Complexity model and complexity estimates. In this section we describe our computational model and briefly mention efficient algorithms for some basic specific algebraic tasks.

2.2.1. Complexity model. Algorithms in computational algebraic geometry are usually described using the standard dense (or sparse) complexity model, i.e. encoding multivariate polynomials by means of the vector of all (or of all nonzero) coefficients. Taking into account that a generic n -variate polynomial of degree d has $\binom{d+n}{n} = O(d^n)$ nonzero coefficients, we see that the dense representation of multivariate polynomials requires an exponential size, and their manipulation usually requires an exponential number of arithmetic operations with respect to the parameters d and n . In order to avoid this exponential behavior, we are going to use an alternative encoding of input, output and intermediate results of our computations by means of straight-line programs (cf. [22], [52], [38], [11]). A **straight-line program** β in $\mathbb{Q}(X) := \mathbb{Q}(X_1, \dots, X_n)$ is a finite sequence of rational functions $(f_1, \dots, f_k) \in \mathbb{Q}(X)^k$ such that for $1 \leq i \leq k$, the function f_i is an element of the set $\{X_1, \dots, X_n\}$, or an element of \mathbb{Q} (a *parameter*), or there exist $1 \leq i_1, i_2 < i$ such that $f_i = f_{i_1} \circ_i f_{i_2}$ holds, where \circ_i is one of the arithmetic operations $+, -, \times, \div$. The straight-line program β is called **division-free** if \circ_i is different from \div for $1 \leq i \leq k$. A natural measure of the complexity of β is its **time** or **length** (cf. [8], [48]), which is the total number of arithmetic operations performed during the evaluation process defined by β . We say that the straight-line program β **computes** or **represents** a subset S of $\mathbb{Q}(X)$ if $S \subset \{f_1, \dots, f_k\}$ holds.

Our model of computation is based on the concept of straight-line programs. However, a model of computation consisting *only* of straight-line programs is not expressive enough for our purposes. Therefore we allow our model to include decisions and selections (subject to previous decisions). For this reason we shall also consider **computation trees**, which are straight-line programs with **branchings**. Time of the evaluation of a given computation tree is defined similarly to the case of straight-line programs (see e.g. [55], [11] for more details on the notion of computation trees).

2.2.2. Probabilistic identity testing. A difficult point in the manipulation of multivariate polynomials given by straight-line programs is the so-called **identity testing problem**: given two elements f and g of $\mathbb{C}[X] := \mathbb{C}[X_1, \dots, X_n]$, decide whether f and g represent the same polynomial function on \mathbb{C}^n . Indeed, all known deterministic algorithms solving this problem have complexity at least $\max\{\deg f, \deg g\}^{\Omega(1)}$. In this article we are going to use *probabilistic* algorithms to solve the identity testing problem, based on the following result:

Theorem 2.2 ([34], [49]). *Let f be a nonzero polynomial of $\mathbb{C}[X]$ of degree at most d and let \mathcal{S} be a finite subset of \mathbb{C} . Then the number of zeros of f in \mathcal{S}^n is at most $d(\#\mathcal{S})^{n-1}$.*

For the analysis of our algorithms, we shall interpret the statement of Theorem 2.2 in terms of probabilities. More precisely, given a fix nonzero polynomial f in $\mathbb{C}[X_1, \dots, X_n]$ of degree at most d , we conclude from Theorem 2.2 that the probability of choosing randomly a point $a \in \mathcal{S}^n$ such that $f(a) = 0$ holds is bounded from above by $d/\#\mathcal{S}$ (assuming a uniform distribution of probability on the elements of \mathcal{S}^n).

2.2.3. Basic complexity estimates. In order to estimate the complexity of our procedures we shall frequently use the notation $M(m) := m \log^2 m \log \log m$. Here and in the sequel \log will denote logarithm in base 2. Let R be a commutative ring of characteristic zero with unity. We recall that the number of arithmetic operations in R necessary to compute the multiplication or division with remainder of two univariate polynomials in $R[T]$ of degree at most m is $O(M(m)/\log(m))$ (cf. [56], [5]). Multipoint evaluation and interpolation of univariate polynomials of $R[T]$ of degree m at *invertible* points $a_1, \dots, a_m \in R$ can be performed with $O(M(m))$ arithmetic operations in R (see e.g. [9]).

If $R = k$ is a field, then we shall use algorithms based on the Extended Euclidean Algorithm (EEA for short) in order to compute the gcd or resultant of two univariate polynomials in $k[T]$ of degree at most m with $O(M(m))$ arithmetic operations in k (cf. [56], [5]). We use Padé approximation in order to compute the dense representation of the numerator and denominator of a rational function $f = p/q \in k(T)$ with $\max\{\deg p, \deg q\} \leq m$ from its Taylor series expansion up to order $2m$. This also requires $O(M(m))$ arithmetic operations in k ([56], [5]).

For brevity, we will denote by Ω the exponent that appears in the complexity estimate $O(n^\Omega)$ for the multiplication of two $(n \times n)$ -matrices with coefficients in \mathbb{Q} . We remark that the (theoretical) bound $\Omega < 2.376$ is typically impractical and we prefer to take $\Omega := \log 7 \sim 2.81$ (cf. [5]).

2.3. Geometric solutions. The notion of a geometric solution of an algebraic variety was first introduced in the works of Kronecker and König in the last years of the XIXth century. Nowadays, geometric solutions are widely used in computer algebra as a suitable representation of algebraic varieties, especially in the zero-dimensional case.

Let \overline{K} denote an algebraic closure of a field K of characteristic zero, let $\mathbb{A}^n(\overline{K})$ be the n -dimensional space \overline{K}^n endowed with its Zariski topology, and let $V = \{\xi^{(1)}, \dots, \xi^{(D)}\}$ be a zero-dimensional subvariety of $\mathbb{A}^n(\overline{K})$ defined over K . A **geometric solution** of V consists of

- a linear form $u = u_1X_1 + \dots + u_nX_n \in K[X]$ which separates the points of V , i.e. satisfying $u(\xi^{(i)}) \neq u(\xi^{(k)})$ if $i \neq k$,
- the minimal polynomial $m_u := \prod_{1 \leq i \leq D} (Y - u(\xi^{(i)})) \in K[Y]$ of u in V (where Y is a new variable),
- polynomials $w_1, \dots, w_n \in K[Y]$ with $\deg w_j < D$ for every $1 \leq j \leq n$ satisfying

$$V = \{(w_1(\eta), \dots, w_n(\eta)) \in \overline{K}^n / \eta \in \overline{K}, m_u(\eta) = 0\}.$$

In the sequel, we shall be given a polynomial system $f_1 = \dots = f_n = 0$ of n -variate polynomials of $\mathbb{Q}[X]$ defining a zero-dimensional affine variety $V \subset \mathbb{A}^n := \mathbb{A}^n(\mathbb{C})$. We shall consider the system $f_1 = \dots = f_n = 0$ (symbolically) “solved” if we obtain a geometric solution of V as defined above.

This notion of geometric solution can be extended to equidimensional varieties of positive dimension. For our purposes, it will be sufficient to consider the case of an algebraic curve defined over \mathbb{Q} .

Suppose that we are given a curve $V \subset \mathbb{A}^{n+1}$ defined by polynomials $f_1, \dots, f_n \in \mathbb{Q}[X, T]$. Assume that for each irreducible component C of V , the identity $I(C) \cap \mathbb{Q}[T] = \{0\}$ holds. Let u be a nonzero linear form of $\mathbb{Q}[X]$ and $\pi_u : V \rightarrow \mathbb{A}^2$ the morphism defined by $\pi_u(x, t) := (t, u(x))$. Our assumptions on V imply that the Zariski closure $\overline{\pi_u(V)}$ of the image of V under π_u is a hypersurface of \mathbb{A}^2 defined over \mathbb{Q} . Let Y be a new indeterminate. Then there exists a unique (up to scaling by nonzero elements of \mathbb{Q}) polynomial $M_u \in \mathbb{Q}[T, Y]$ of minimal degree defining $\overline{\pi_u(V)}$. Let $m_u \in \mathbb{Q}(T)[Y]$ denote the (unique) monic multiple of M_u with $\deg_Y(m_u) = \deg_Y(M_u)$. We call m_u the *minimal polynomial* of u in V . In these terms, a **geometric solution** of the curve V consists of

- a *generic* linear form $u \in \mathbb{Q}[X]$,
- the minimal polynomial $m_u \in \mathbb{Q}(T)[Y]$,
- elements v_1, \dots, v_n of $\mathbb{Q}(T)[Y]$ such that $\frac{\partial m_u}{\partial Y} X_i \equiv v_i \pmod{\mathbb{Q}(T) \otimes \mathbb{Q}[V]}$ and $\deg_Y(v_i) < \deg_Y(m_u)$ holds for $1 \leq i \leq n$.

We observe that $\deg_Y m_u$ equals the cardinality of the zero-dimensional variety defined by f_1, \dots, f_n over $\mathbb{A}^n(\overline{\mathbb{Q}(T)})$.

In the sequel, we shall deal with curves $V := V(f_1, \dots, f_n) \subset \mathbb{A}^{n+1}$ as above. The complexity of the algorithms for solving the systems $f_1 = \dots = f_n = 0$ defining such curves will be expressed mainly by means of two discrete invariants: the *degree* and the *height* of the projection $\pi : V \rightarrow \mathbb{A}^1$. The degree of π is defined as the degree $\deg m_u = \deg_Y M_u$ of the minimal polynomial of a generic linear form $u \in \mathbb{Q}[X_1, \dots, X_n]$ and can be considered as a measure of the “complexity” of the curve V . On the other hand, the height of π is defined as $\deg_T M_u$ and may be considered as a measure of the “complexity of the description” of the curve V .

In the sparse setting, we can estimate $\deg_Y M_u$ and $\deg_T M_u$ in terms of combinatorial quantities (namely, mixed volumes) associated to the polynomial system under consideration (see also [43]).

Lemma 2.3. *Let assumptions and notations be as above. For $1 \leq i \leq n$, let $Q_i \subset \mathbb{R}^n$ be the Newton polytope of f_i , considering f_i as an element of $\mathbb{Q}(T)[X]$. Let $\hat{Q}_1, \dots, \hat{Q}_n \subset \mathbb{R}^{n+1}$ be the Newton polytopes of f_1, \dots, f_n , considering f_1, \dots, f_n as elements of $\mathbb{Q}[X, T]$, and let $\Delta \subset \mathbb{R}^{n+1}$ be the standard unitary simplex in the plane $\{T = 0\}$, i.e., the Newton polytope of a generic linear form $u \in \mathbb{Q}[X]$. Assume*

that $0 \in \widehat{Q}_i$ for every $1 \leq i \leq n$. Then the following estimates hold:

$$(2.2) \quad \deg_Y M_u \leq MV_n(Q_1, \dots, Q_n), \quad \deg_T M_u \leq MV_n(\Delta, \widehat{Q}_1, \dots, \widehat{Q}_n).$$

Furthermore, if there exist $c_1, \dots, c_n \in \mathbb{R}_{\geq 0}$ such that $\widehat{Q}_i \subset Q_i \times [0, c_i]$ for $1 \leq i \leq n$, then the following inequality holds:

$$(2.3) \quad \deg_T M_u \leq \sum_{i=1}^n c_i MV(\Delta, Q_1, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_n).$$

Proof. The upper bound $\deg_Y M_u \leq MV_n(Q_1, \dots, Q_n)$ follows straightforwardly from the BKK bound and the affine root count in [33].

In order to obtain an upper bound for $\deg_T M_u$, we observe that substituting a generic value $y \in \mathbb{Q}$ for Y we have $\deg_T M_u(T, Y) = \deg_T M_u(T, y) = \#\{t \in \mathbb{C}; M_u(t, y) = 0\}$. Moreover, it follows that $M_u(t, y) = 0$ if and only if there exists a point $x \in \mathbb{A}^n$ with $y = u(x)$ and $(x, t) \in V$. Thus, it suffices to estimate the number of points $(x, t) \in \mathbb{A}^{n+1}$ satisfying $u(x) - y = 0, f_1(x, t) = 0, \dots, f_n(x, t) = 0$. Being u a generic linear form, the system

$$(2.4) \quad u(X) - y = 0, f_1(X, T) = 0, \dots, f_n(X, T) = 0$$

has finitely many common zeros in \mathbb{A}^{n+1} . Combining the BKK bound with the affine root count of [33] we see that there are at most $MV(\Delta, \widehat{Q}_1, \dots, \widehat{Q}_n)$ solutions of (2.4). We conclude that $\deg_T M_u \leq MV(\Delta, \widehat{Q}_1, \dots, \widehat{Q}_n)$ holds, which shows (2.2).

In order to prove (2.3), we make use of basic properties of the mixed volume (see, for instance, [17, Ch. IV]). Since $\widehat{Q}_i \subset Q_i \times [0, c_i]$ holds for $1 \leq i \leq n$, by the monotonicity of the mixed volume we have

$$MV(\Delta, \widehat{Q}_1, \dots, \widehat{Q}_n) \leq MV(\Delta, Q_1 \times [0, c_1], \dots, Q_n \times [0, c_n]).$$

Note that $Q_i \times [0, c_i] = S_{i,0} + S_{i,1}$, where $S_{i,0} = Q_i \times \{0\}$ and $S_{i,1} = \{0\} \times [0, c_i]$ for $i = 1, \dots, n$. Hence, by multilinearity,

$$(2.5) \quad MV(\Delta, Q_1 \times [0, c_1], \dots, Q_n \times [0, c_n]) = \sum_{(j_1, \dots, j_n) \in \{0,1\}^n} MV(\Delta, S_{1,j_1}, \dots, S_{n,j_n}).$$

If the vector (j_1, \dots, j_n) has at least two nonzero coordinates, then two of the sets $S_{1,j_1}, \dots, S_{n,j_n}$ are parallel line segments; therefore, $MV(\Delta, S_{1,j_1}, \dots, S_{n,j_n}) = 0$. On the other hand, if j_i is the only nonzero coordinate, the corresponding term in the sum of the right-hand side of (2.5) is

$$\begin{aligned} MV_{n+1}(\Delta, Q_1 \times \{0\}, \dots, Q_{i-1} \times \{0\}, \{0\} \times [0, c_i], Q_{i+1} \times \{0\}, \dots, Q_n \times \{0\}) \\ = c_i MV_n(\Delta, Q_1, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_n). \end{aligned}$$

Finally, for $(j_1, \dots, j_n) = (0, \dots, 0)$ we have $MV_{n+1}(\Delta, Q_1 \times \{0\}, \dots, Q_n \times \{0\}) = 0$ since all the polytopes are included in an n -dimensional subspace.

We conclude that the right-hand side of (2.5) equals the right-hand side of (2.3). This finishes the proof of the lemma. \square

From the algorithmic point of view, the crucial step towards the computation of a geometric solution of the variety V consists in the computation of the minimal polynomial m_u of a generic linear form u which separates the points of V . In the remaining part of this section we shall show how we can derive an algorithm for computing the entire geometric solution of a zero-dimensional variety V defined

over \mathbb{Q} from a procedure for computing the minimal polynomial of a generic linear form u (cf. [2], [21], [50]).

Let $\Lambda := (\Lambda_1, \dots, \Lambda_n)$ be a vector of new indeterminates and let $K := \mathbb{Q}(\Lambda)$. Denote by I_K the ideal in $K[X_1, \dots, X_n]$ which is the extension of the ideal $I := I(V) \subset \mathbb{Q}[X_1, \dots, X_n]$ of the zero-dimensional variety V , and denote by $B := K[X_1, \dots, X_n]/I_K$ the corresponding zero-dimensional quotient algebra. Write $V = \{\xi^{(1)}, \dots, \xi^{(D)}\}$.

Set $U := \Lambda_1 X_1 + \dots + \Lambda_n X_n \in K[X_1, \dots, X_n]$ and let $m_U(\Lambda, Y) = \prod_{j=1}^D (Y - U(\xi^{(j)})) \in \mathbb{Q}[\Lambda, Y]$ be the minimal polynomial of the linear form U in the extension $K \hookrightarrow B$. Note that $\deg m_U = D$ holds, and that $\partial m_U / \partial Y$ is not a zero divisor in $\mathbb{Q}[\Lambda^n \times V]$. Furthermore, $m_U(\Lambda, U) \in I(\mathbb{A}^n \times V) \subset \mathbb{Q}[\Lambda, X_1, \dots, X_n]$ holds. Since $I(\mathbb{A}^n \times V)$ is generated by polynomials in $\mathbb{Q}[X_1, \dots, X_n]$, taking the partial derivative of $m_U(\Lambda, U)$ with respect to the variable Λ_k for $1 \leq k \leq n$, we conclude that

$$(2.6) \quad \frac{\partial m_U}{\partial Y}(\Lambda, U) X_k + \frac{\partial m_U}{\partial \Lambda_k}(\Lambda, U) \in I(\mathbb{A}^n \times V).$$

Observe that the degree estimate $\deg_Y(\partial m_U / \partial \Lambda_k) \leq D - 1$ holds.

Assume that a linear form $u = u_1 X_1 + \dots + u_n X_n \in \mathbb{Q}[X_1, \dots, X_n]$ which separates the points of V is given. Substituting u_k for Λ_k in the polynomial $m_U(\Lambda, Y)$ we obtain the minimal polynomial $m_u(Y)$ of u . Furthermore, making the same substitution in the polynomials $(\partial m_U / \partial Y)(\Lambda, Y) X_k + (\partial m_U / \partial \Lambda_k)(\Lambda, Y)$ of (2.6) for $1 \leq k \leq n$ and reducing modulo $m_u(Y)$, we obtain polynomials $(\partial m_u / \partial Y)(Y) X_k - v_k(Y) \in I(V)$ ($1 \leq k \leq n$). In particular, we have that the identities

$$(2.7) \quad \frac{\partial m_u}{\partial Y}(u) X_k = v_k(u) \quad (1 \leq k \leq n)$$

hold in $\mathbb{Q}[V]$. Observe that the minimal polynomial $m_u(Y)$ is square-free, since the linear form u separates the points of V . Therefore, $m_u(Y)$ and $\partial m_u / \partial Y(Y)$ are relatively prime. Thus, multiplying modulo $m_u(Y)$ the polynomials $v_k(Y)$ by the inverse of $(\partial m_u / \partial Y)(Y)$ modulo $m_u(Y)$ we obtain polynomials $w_k(Y) := (\partial m_u / \partial Y)^{-1} v_k(Y)$ ($1 \leq k \leq n$) of degree at most $D - 1$ such that

$$(2.8) \quad X_k = w_k(u)$$

holds in $\mathbb{Q}[V]$ for $1 \leq k \leq n$. The polynomials $m_u, w_1, \dots, w_n \in \mathbb{Q}[Y]$ form a geometric solution of V .

Now, suppose that we are given an algorithm Ψ over $\mathbb{Q}(\Lambda)$ for computing the minimal polynomial of the linear form $U = \Lambda_1 X_1 + \dots + \Lambda_n X_n$. Suppose further that we are given a separating linear form $u := u_1 X_1 + \dots + u_n X_n \in \mathbb{Q}[X_1, \dots, X_n]$ such that the vector (u_1, \dots, u_n) does not annihilate any denominator in $\mathbb{Q}[\Lambda]$ of any intermediate result of the algorithm Ψ . In order to compute the polynomials v_1, \dots, v_n of (2.7), we observe that the Taylor expansion of $m_U(\Lambda, Y)$ in powers of $\Lambda - u := (\Lambda_1 - u_1, \dots, \Lambda_n - u_n)$ has the following expression:

$$m_U(\Lambda, Y) = m_u(Y) + \sum_{k=1}^n \left(\frac{\partial m_u}{\partial Y}(Y) X_k - v_k(Y) \right) (\Lambda_k - u_k) \pmod{(\Lambda - u)^2}.$$

We shall compute this first-order Taylor expansion by computing the first-order Taylor expansion of each intermediate result in the algorithm Ψ . In this way, each arithmetic operation in $\mathbb{Q}(\Lambda)$ arising in the algorithm Ψ becomes an arithmetic

operation between two polynomials of $\mathbb{Q}[\Lambda]$ of degree at most 1, and is truncated up to order $(\Lambda - u)^2$. Since the first-order Taylor expansion of an addition, multiplication or division of two polynomials of $\mathbb{Q}[\Lambda]$ of degree at most 1 requires $O(n)$ arithmetic operations in \mathbb{Q} , we have that the whole step requires $O(nT)$ arithmetic operations in \mathbb{Q} , where T is the number of arithmetic operations in $\mathbb{Q}(\Lambda)$ performed by the algorithm Ψ .

Finally, the computation of the polynomials w_1, \dots, w_n of (2.8) requires the inversion of $\partial m_u / \partial Y$ modulo $m_u(Y)$ and the modular multiplication $w_k(Y) := (\partial m_u / \partial Y)^{-1} v_k(Y)$ for $1 \leq k \leq n$. These steps can be executed with additional $O(nM(D))$ arithmetic operations in \mathbb{Q} . Summarizing, we have the following result:

Lemma 2.4. *Suppose that we are given:*

- (1) *an algorithm Ψ in $\mathbb{Q}(\Lambda)$ which computes the minimal polynomial $m_U \in \mathbb{Q}[\Lambda, Y]$ of $U := \Lambda X_1 + \dots + \Lambda_n X_n$ with T arithmetic operations in $\mathbb{Q}(\Lambda)$,*
- (2) *a separating linear form $u := u_1 X_1 + \dots + u_n X_n \in \mathbb{Q}[X_1, \dots, X_n]$ such that the vector (u_1, \dots, u_n) does not annihilate any denominator in $\mathbb{Q}[\Lambda]$ of any intermediate result of the algorithm Ψ .*

Then a geometric solution of the variety V can be (deterministically) computed with $O(n(T + M(D)))$ arithmetic operations in \mathbb{Q} .

3. STATEMENT OF THE PROBLEM AND OUTLINE OF THE ALGORITHM

Let $\Delta_1, \dots, \Delta_n$ be fixed finite subsets of $\mathbb{Z}_{\geq 0}^n$ with $0 \in \Delta_i$ for $1 \leq i \leq n$ and let $D := MV(Q_1, \dots, Q_n)$ denote the mixed volume of the polytopes $Q_1 := \text{conv}(\Delta_1), \dots, Q_n := \text{conv}(\Delta_n)$. Assume that $D > 0$ holds or, equivalently, that $\dim(\sum_{i \in I} Q_i) \geq |I|$ for every non-empty subset $I \subset \{1, \dots, n\}$ (see, for instance, [37, Chapter IV, Proposition 2.3]).

Let $f_1, \dots, f_n \in \mathbb{Q}[X]$ be polynomials defining a sparse system with respect to $\Delta_1, \dots, \Delta_n$ and let d_1, \dots, d_n be their total degrees. Let $d := \{d_1, \dots, d_n\}$. Suppose that f_1, \dots, f_n define a zero-dimensional variety V in \mathbb{A}^n . As in the previous section, we group equal supports into $s \leq n$ distinct supports $\mathcal{A}^{(1)}, \dots, \mathcal{A}^{(s)}$. Write $\mathcal{A} := (\mathcal{A}^{(1)}, \dots, \mathcal{A}^{(s)})$ and denote by k_ℓ the number of polynomials f_i with support $\mathcal{A}^{(\ell)}$ for $1 \leq \ell \leq s$.

From now on we assume that we are given a sufficiently generic lifting function $\omega := (\omega_1, \dots, \omega_s)$ and the fine-mixed subdivision of \mathcal{A} induced by ω . We assume further that the function $\omega_\ell : \mathcal{A}^{(\ell)} \rightarrow \mathbb{Z}$ takes only nonnegative values and $\omega_\ell(0, \dots, 0) = 0$ for every $1 \leq \ell \leq s$. The lifting function ω and the corresponding fine-mixed subdivision of \mathcal{A} can be used in order to define an appropriate deformation of the input system, the so-called *polyhedral deformation* introduced by Huber and Sturmfels in [25]. Our purpose here is to use this polyhedral deformation to derive a symbolic probabilistic algorithm which computes a geometric solution of the input system $f_1 = 0, \dots, f_n = 0$.

Since the polyhedral deformation requires that the coefficients of the input polynomials satisfy certain generic conditions, we introduce some auxiliary *generic* polynomials g_1, \dots, g_n with the same supports $\Delta_1, \dots, \Delta_n$ and consider the perturbed polynomial system defined by $h_i := f_i + g_i$ for $1 \leq i \leq n$. The genericity conditions underlying the choice of g_1, \dots, g_n and h_1, \dots, h_n are discussed in Section 3.1. We observe that if the coefficients of the input polynomials f_1, \dots, f_n satisfy these conditions then our method can be directly applied to f_1, \dots, f_n .

Otherwise, we first solve the system $h_1 = 0, \dots, h_n = 0$ and then recover the solutions to the input system $f_1 = 0, \dots, f_n = 0$ by considering the homotopy $f_1 + (1 - T)g_1 = \dots = f_n + (1 - T)g_n = 0$.

3.1. The polyhedral deformation. This section is devoted to introducing the polyhedral deformation of Huber and Sturmfels.

Let $h_i := \sum_{q \in \Delta_i} c_{i,q} X^q$ for $1 \leq i \leq n$ be polynomials in $\mathbb{Q}[X]$ and let V_1 denote the set of their common zeros in \mathbb{A}^n . For $i = 1, \dots, n$, let ℓ_i be the (unique) integer with $\Delta_i = \mathcal{A}^{(\ell_i)}$, and let $\tilde{\omega}_i := \omega_{\ell_i}$ be the lifting function associated to the support Δ_i . In order to simplify notations, the n -tuple $\tilde{\omega} := (\tilde{\omega}_1, \dots, \tilde{\omega}_n)$ will be denoted simply by $\omega = (\omega_1, \dots, \omega_n)$. As before, we denote by $\hat{C}^{(\ell)} := \{(q, \omega_\ell(q)) \in \mathbb{R}^{n+1} : q \in C^{(\ell)}\}$ the graph of any subset $C^{(\ell)}$ of $\mathcal{A}^{(\ell)}$ for $1 \leq \ell \leq s$, and extend this notation correspondingly. For a new indeterminate T , we deform the polynomials h_1, \dots, h_n into polynomials $\hat{h}_1, \dots, \hat{h}_n \in \mathbb{Q}[X, T]$ defined in the following way:

$$(3.1) \quad \hat{h}_i(X, T) := \sum_{q \in \Delta_i} c_{i,q} X^q T^{\omega_i(q)} \quad (1 \leq i \leq n).$$

Let I denote the ideal of $\mathbb{Q}[X, T]$ generated by $\hat{h}_1, \dots, \hat{h}_n$ and let J denote the Jacobian determinant of $\hat{h}_1, \dots, \hat{h}_n$ with respect to the variables X_1, \dots, X_n . We set

$$(3.2) \quad \hat{V} := V(I : J^\infty) \subset \mathbb{A}^{n+1}.$$

We shall show that, under a generic choice of the coefficients of h_1, \dots, h_n , the system defined by the polynomials in (3.1) constitutes a deformation of the input system $h_1 = 0, \dots, h_n = 0$, in the sense that the morphism $\pi : \hat{V} \rightarrow \mathbb{A}^1$ defined by $\pi(x, t) := t$ is a dominant map with $\pi^{-1}(1) = V_1 \times \{1\}$. We shall further exhibit degree estimates on the genericity condition underlying such choice of coefficients. These estimates will allow us to obtain suitable polynomials h_1, \dots, h_n by randomly choosing their coefficients in an appropriate finite subset of \mathbb{Z} .

According to [25, Section 3], the solutions over an algebraic closure $\overline{\mathbb{Q}(T)}$ of $\mathbb{Q}(T)$ to the system defined by the polynomials (3.1) are algebraic functions of the parameter T which can be represented as Puiseux series of the form

$$(3.3) \quad x(T) := (x_{10} T^{\gamma_1/\gamma_{n+1}} + \text{h-o t.}, \dots, x_{n0} T^{\gamma_n/\gamma_{n+1}} + \text{h-o t.}),$$

where $\gamma := (\gamma_1, \dots, \gamma_n, \gamma_{n+1}) \in \mathbb{Z}^{n+1}$ is an inner normal with positive last coordinate $\gamma_{n+1} > 0$ of a (lower) facet $\hat{C} = (\hat{C}^{(1)}, \dots, \hat{C}^{(s)})$ of $\hat{\mathcal{A}}$ of type (k_1, \dots, k_s) , and $x_0 := (x_{10}, \dots, x_{n0}) \in (\mathbb{C}^*)^n$ is a solution to the polynomial system defined by

$$(3.4_\gamma) \quad h_{i,\gamma}^{(0)} := \sum_{q \in C^{(\ell_i)}} c_{i,q} X^q \quad (1 \leq i \leq n),$$

where, as defined before, ℓ_i is the integer with $1 \leq \ell_i \leq s$ and $\Delta_i = \mathcal{A}^{(\ell_i)}$. We shall “lift” each solution x_0 to this system to a solution of the form (3.3) to the system defined by (3.1). This means that, on input x_0 , we shall compute the Puiseux series expansion of the corresponding solution (3.3) truncated up to a suitable order.

Let

$$(3.5) \quad V_{0,\gamma} := \{x \in (\mathbb{C}^*)^n : h_{1,\gamma}^{(0)}(x) = 0, \dots, h_{n,\gamma}^{(0)}(x) = 0\}.$$

A particular feature of the polynomials (3.4_γ) which makes the associated equation system “easy to solve” is that the vector of their supports is $(C^{(1)})^{k_1} \times \dots \times (C^{(s)})^{k_s}$,

where $(C^{(1)}, \dots, C^{(s)})$ is a cell of type (k_1, \dots, k_s) in a fine-mixed subdivision of \mathcal{A} . Therefore, for every $1 \leq \ell \leq s$, the set $C^{(\ell)}$ consists of $k_\ell + 1$ points and hence, up to monomial multiplication so that each polynomial has a non-zero constant term, the (Laurent) polynomials in (3.4 _{γ}) are linear combinations of $n + 1$ distinct monomials in n variables.

Denote $\Gamma \subset \mathbb{Z}^{n+1}$ the set of all primitive integer vectors of the form $\gamma := (\gamma_1, \dots, \gamma_n, \gamma_{n+1}) \in \mathbb{Z}^{n+1}$ with $\gamma_{n+1} > 0$ for which there is a cell $C = (C^{(1)}, \dots, C^{(s)})$ of type (k_1, \dots, k_s) of the subdivision of \mathcal{A} induced by ω such that \widehat{C} has inner normal γ .

Fix a cell $C = (C^{(1)}, \dots, C^{(s)})$ of type (k_1, \dots, k_s) of the subdivision of \mathcal{A} induced by ω associated with a primitive inner normal $\gamma \in \Gamma$ with positive last coordinate. In order to lift the points of the variety $V_{0,\gamma}$ of (3.5) to a solution of the system defined by the polynomials in (3.1), we will work with a family of auxiliary polynomials $h_{1,\gamma}, \dots, h_{n,\gamma} \in \mathbb{Q}[X, T]$ which we define as follows:

$$(3.6) \quad h_{i,\gamma}(X, T) := T^{-m_i} \widehat{h}_i(T^{\gamma_1} X_1, \dots, T^{\gamma_n} X_n, T^{\gamma_{n+1}}) \quad (1 \leq i \leq n)$$

where $m_i \in \mathbb{Z}$ is the lowest power of T appearing in $\widehat{h}_i(T^{\gamma_1} X_1, \dots, T^{\gamma_n} X_n, T^{\gamma_{n+1}})$ for every $1 \leq i \leq n$. Note that the polynomials obtained by substituting $T = 0$ into $h_{1,\gamma}, \dots, h_{n,\gamma}$ are precisely those introduced in (3.4 _{γ}).

3.2. On the genericity of the initial system. Here we discuss the genericity conditions underlying the choice of the polynomials g_1, \dots, g_n that enable us to apply the polyhedral deformation defined by the lifting form ω to the system $h_1 := f_1 + g_1 = 0, \dots, h_n := f_n + g_n = 0$.

The first condition we require is that the set of common zeros of the perturbed polynomials h_1, \dots, h_n is a zero-dimensional variety with the maximum number of points for a sparse system with the given structure. More precisely, we require the following condition:

- (H1) The set $V_1 := \{x \in \mathbb{A}^n : h_1(x) = 0, \dots, h_n(x) = 0\}$ is a zero-dimensional variety with $D := MV(Q_1, \dots, Q_n)$ distinct points.

In addition, we need that the system (3.4 _{γ}) giving the initial points to our first deformation for every $\gamma \in \Gamma$ has as many roots as possible, namely the mixed volume of their support vectors.

For each cell $C = (C^{(1)}, \dots, C^{(s)})$ of type (k_1, \dots, k_s) of the induced fine-mixed subdivision, set an order on the $n + 1$ points appearing in any of the sets $C^{(\ell)}$, after a suitable translation so that $0 \in C^{(\ell)}$ for every $1 \leq \ell \leq s$. Assume that $0 \in \mathbb{Z}^n$ is the last point according to this order. Denote $\gamma \in \mathbb{Z}^{n+1}$ the primitive inner normal of C with positive last coordinate. Consider the $n \times (n + 1)$ matrix whose i th row is the coefficient vector of $h_{i,\gamma}^{(0)}$ in the prescribed monomial order and set $\mathcal{M}_\gamma \in \mathbb{Q}^{n \times n}$ and $\mathcal{B}_\gamma \in \mathbb{Q}^{n \times 1}$ for the submatrices consisting of the first n columns (coefficients of non-constant monomials) and the last column (constant coefficients) respectively. Then, the coefficients of g_1, \dots, g_n are to be chosen so that the following condition holds:

- (H2) For every $\gamma \in \Gamma$, the $(n \times n)$ -matrix \mathcal{M}_γ is nonsingular and all the entries of $(\mathcal{M}_\gamma)^{-1} \mathcal{B}_\gamma$ are nonzero.

Our next results assert that the above conditions can be met with good probability by randomly choosing the coefficients of g_1, \dots, g_n in a certain set $\mathcal{S} \subset \mathbb{Z}$. We

observe that our estimate on the size of \mathcal{S} is not intended to be accurate, but to show that the growth of the size of the integers involved in the subsequent computations is not likely to create complexity problems.

Let $\{\Omega_{i,q} : 1 \leq i \leq n, q \in \Delta_i\}$ be a set of new indeterminates over \mathbb{Q} . For $1 \leq i \leq n$, write $\Omega_i := (\Omega_{i,q} : q \in \Delta_i)$ and let $H_i \in \mathbb{Q}[\Omega_i, X]$ be the generic polynomial

$$(3.7) \quad H_i(\Omega_i, X) := \sum_{q \in \Delta_i} \Omega_{i,q} X^q$$

with support Δ_i and $N_i := \#\Delta_i$ coefficients. Let $\Omega := (\Omega_1, \dots, \Omega_n)$ and let $N := N_1 + \dots + N_n$ be the total number of indeterminate coefficients.

We start the analysis of the required generic conditions with the following quantitative version of Bernstein's result on the genericity of zero-dimensional sparse systems (see [4, Theorem B], [25, Theorem 6.1]):

Lemma 3.1. *There exists a nonzero polynomial $P^{(0)} \in \mathbb{Q}[\Omega]$ with $\deg P^{(0)} \leq 3n^{2n+1}d^{2n-1}$ such that for any $c \in \mathbb{Q}^N$ with $P^{(0)}(c) \neq 0$, the system $H_1(c_1, X) = 0, \dots, H_n(c_n, X) = 0$ has D solutions in \mathbb{A}^n , counting multiplicities.*

Proof. Due to [25, Theorem 6.1] combined with [33], the system $H_1(c_1, X) = 0, \dots, H_n(c_n, X) = 0$ has D solutions in \mathbb{A}^n counting multiplicities if and only if for every facet inner normal $\mu \in \mathbb{Z}^n$ of $Q_1 + \dots + Q_n$, the sparse resultant $\text{Res}_{\Delta_1^\mu, \dots, \Delta_n^\mu}$ does not vanish at $c := (c_1, \dots, c_n)$. Here Δ_i^μ denotes the set of points of Δ_i where the linear functional induced by μ attains its minimum for $1 \leq i \leq n$.

Therefore, the polynomial $P^{(0)} := \prod_{\mu} \text{Res}_{\Delta_1^\mu, \dots, \Delta_n^\mu} \in \mathbb{Q}[\Omega]$, where the product ranges over all primitive inner normals $\mu \in \mathbb{Z}^n$ to facets of $Q_1 + \dots + Q_n$, satisfies the required condition.

In order to estimate the degree of $P^{(0)}$, we observe that for every facet inner normal $\mu \in \mathbb{Z}^n$ the following upper bound holds:

$$\deg(\text{Res}_{\Delta_1^\mu, \dots, \Delta_n^\mu}) \leq \sum_{i=1}^n MV(\Delta_1^\mu, \dots, \widehat{\Delta_i^\mu}, \dots, \Delta_n^\mu) \leq nd^{n-1},$$

where $d := \max\{d_1, \dots, d_n\}$. On the other hand, it is not difficult to see that the number of facets of an n -dimensional integer convex polytope $P \subset \mathbb{R}^n$ which has an integer point in its interior is bounded by $n! \text{vol}_{\mathbb{R}^n}(P)$. Now, taking $P := (n+1)Q$, we obtain an integer polytope with the same number of facets as Q having an integer interior point. Then, the number of facets of Q is bounded by $n! \text{vol}_{\mathbb{R}^n}(P) = n! \text{vol}_{\mathbb{R}^n}((n+1)Q) = (n+1)^n n! \text{vol}_{\mathbb{R}^n}(Q) \leq (n+1)^n (nd)^n$, since Q is included in the n -dimensional simplex of size nd . This proves the upper bound for the degree $P^{(0)}$ of the statement of the lemma. \square

The next lemma is concerned with the genericity of a smooth sparse system.

Lemma 3.2. *With the same notations as in Lemma 3.1 and before, there exists a nonzero polynomial $P^{(1)} \in \mathbb{Q}[\Omega]$ of degree at most $4n^{2n+1}d^{2n-1}$ such that for any $c \in \mathbb{Q}^N$ with $P^{(1)}(c) \neq 0$, the system $H_1(c_1, X) = 0, \dots, H_n(c_n, X) = 0$ has exactly D distinct solutions in \mathbb{A}^n .*

Proof. Consider the incidence variety associated to $(\Delta_1, \dots, \Delta_n)$ -sparse systems, namely

$$W := \{(x, c) \in (\mathbb{C}^*)^n \times (\mathbb{A}^{N_1} \times \dots \times \mathbb{A}^{N_n}) : \sum_{q \in \Delta_i} c_{i,q} x^q = 0 \text{ for } 1 \leq i \leq n\}.$$

As in [40, Proposition 2.3], it follows that W is a \mathbb{Q} -irreducible variety. Let $\pi_\Omega : W \rightarrow \mathbb{A}^{N_1} \times \dots \times \mathbb{A}^{N_n}$ be the canonical projection, which is a dominant map.

By [37, Chapter V, Corollary (3.2.1)], there is a nonempty Zariski open set $\mathcal{U}(\Delta_1, \dots, \Delta_n) \subset \mathbb{A}^{N_1} \times \dots \times \mathbb{A}^{N_n}$ of coefficients $c = (c_1, \dots, c_n)$ for which the polynomials $H_1(c_1, X), \dots, H_n(c_n, X)$ have supports $\Delta_1, \dots, \Delta_n$ respectively and the set of their common zeros in $(\mathbb{C}^*)^n$ is a non-degenerate complete intersection variety. Then, the Jacobian $J_H := \det(\partial H_i / \partial X_j)_{1 \leq i, j \leq n}$ does not vanish at any point of $\pi_\Omega^{-1}(c)$ for every $c \in \mathcal{U}(\Delta_1, \dots, \Delta_n)$.

Let $\mathbb{Q}(\Omega) \hookrightarrow \mathbb{Q}(W)$ be the finite field extension induced by the dominant projection π_Ω . By the preceding paragraph we have that the rational function defined by J_H in $\mathbb{Q}(W)$ is nonzero. Therefore, its primitive minimal polynomial $M_J \in \mathbb{Q}[\Omega, T]$ is well defined and satisfies the degree estimates

$$\deg_\Omega M_J \leq \deg W \cdot \deg J_H \leq \prod_{i=1}^n (d_i + 1) \cdot \sum_{i=1}^n d_i \leq 2^n d^{n+1} n$$

(see [47], [50]).

Let $P^{(1)} := P^{(0)} M_J^{(0)}$, where $P^{(0)}$ is the polynomial given by Lemma 3.1 and $M_J^{(0)}$ denotes the constant term of the expansion of M_J in powers of T . We claim that $P^{(1)}$ satisfies the requirements of the statement of the lemma. Indeed, let $c \in \mathbb{Q}^N$ satisfy $P^{(1)}(c) \neq 0$. Then $P^{(0)}(c) \neq 0$ holds and so, Lemma 3.1 implies that $H_1(c, X) = \dots = H_n(c, X) = 0$ is a zero-dimensional system. Furthermore, $M_J^{(0)}(c)$ is a nonzero multiple of the product $\prod_{x \in \pi_\Omega^{-1}(c)} J_H(c, x)$. Thus, the non-vanishing of $M_J^{(0)}(c)$ shows that all the points of $\pi_\Omega^{-1}(c)$ are smooth and therefore, from e.g. [37, IV, Theorem 2.2], it follows that $\pi_\Omega^{-1}(c)$ consists of exactly D simple points in $(\mathbb{C}^*)^n$. Moreover, combining the assumption that $0 \in \Delta_i$ for $1 \leq i \leq n$ with [33, Theorem 2.4], we deduce that $\pi_\Omega^{-1}(c)$ consists of D simple points in \mathbb{A}^n . The estimate $\deg M_J^{(0)} \leq \deg_\Omega M_J \leq 2^n d^{n+1} n \leq n^{2(n+1)} d^{2n-1}$ implies the statement of the lemma. \square

Finally, we exhibit a generic condition on the coefficients h_1, \dots, h_n which implies that assumption (H2) holds.

Lemma 3.3. *With the previous assumptions and notations, there exists a nonzero polynomial $P^{(2)} \in \mathbb{Q}[\Omega]$ with $\deg P^{(2)} \leq n(n+1)\#\Gamma$ such that for every $c := (c_1, \dots, c_n) \in \mathbb{Q}^N$ with $P^{(2)}(c) \neq 0$, the polynomials $h_i := H_i(c_i, X)$ ($1 \leq i \leq n$) satisfy condition (H2).*

Proof. Fix a primitive integer inner normal $\gamma \in \Gamma$ to a lower facet of $\widehat{\mathcal{A}}$. Let $\mathcal{M}_\gamma \in \mathbb{Q}[\Omega]^{n \times n}$ and $\mathcal{B}_\gamma \in \mathbb{Q}[\Omega]^{n \times 1}$ be the matrices constructed from the generic polynomials $H_1, \dots, H_n \in \mathbb{Q}[\Omega][X]$ as explained in the paragraph preceding condition (H2). Let $D_{0,\gamma} \in \mathbb{Q}[\Omega]$ be the (non-zero) determinant of \mathcal{M}_γ , and for every $1 \leq j \leq n$, let $D_{j,\gamma}$ be the determinant of the matrix obtained from \mathcal{M}_γ by replacing its j th column with \mathcal{B}_γ . Set $P_\gamma := \prod_{j=0}^n D_{j,\gamma}$. Finally, take $P^{(2)} := \prod_{\gamma \in \Gamma} P_\gamma$.

By Cramer's rule, whenever $P^{(2)}(c) \neq 0$, we have that the system h_1, \dots, h_n with coefficient vector $c = (c_1, \dots, c_n)$ meets condition (H2).

The degree estimate for $P^{(2)}$ follows from the fact that $\deg P_\gamma \leq n(n+1)$ holds for every $\gamma \in \Gamma$, since each of the entries of the matrices whose determinants are involved has degree 1 in the variables Ω . \square

Now, we are ready to state a generic condition on the coefficients of h_1, \dots, h_n which implies that (H1) and (H2) hold.

Proposition 3.4. *Under the previous assumptions and notations, there exists a nonzero polynomial $P \in \mathbb{Q}[\Omega]$ with $\deg P \leq 4n^{2n+1}d^{2n-1} + n(n+1)D$ such that for every $c \in \mathbb{Q}^N$ with $P(c) \neq 0$, the polynomials $h_i := H_i(c_i, X)$ ($1 \leq i \leq n$) satisfy conditions (H1) and (H2).*

Proof. Set $P := P^{(1)}P^{(2)}$, where $P^{(1)}$ is the polynomial of the statement of Lemma 3.2 and $P^{(2)}$ is the one defined in the statement of Lemma 3.3. The result follows from Lemmas 3.2 and 3.3, and the upper bound $\#\Gamma \leq D$ for the cardinality of the set of the distinct inner normal vectors considered (one for each cell of type (k_1, \dots, k_s) in the given fine-mixed subdivision). \square

3.3. Outline of the algorithm. Now we have all the tools necessary to give an outline of our algorithm for the computation of a geometric solution of the (sufficiently generic) sparse system $h_1 = \dots = h_n = 0$.

With notations as in the previous subsections, we assume that a fine-mixed subdivision of \mathcal{A} induced by a lifting function ω is given. This means that we are given the set Γ of inner normals of the lower facets of the convex hull of $\hat{\mathcal{A}}$, together with the corresponding cells of the convex hull of \mathcal{A} . In addition, we suppose that our input polynomials $h_1, \dots, h_n \in \mathbb{Q}[X]$ satisfy conditions (H1) and (H2) and denote by $V_1 \subset \mathbb{A}^n$ the affine variety defined by h_1, \dots, h_n .

First, we choose a *generic* linear form $u \in \mathbb{Q}[X]$ such that:

- u separates the points of the zero-dimensional varieties V_1 and $V_{0,\gamma}$ for every $\gamma \in \Gamma$. This condition is represented by the nonvanishing of a certain nonconstant polynomial of degree at most $2D^2$.
- An algorithm for the computation of the minimal polynomial of u in $V_{0,\gamma}$ to be described below can be extended to a computation of a geometric solution of $V_{0,\gamma}$ according to Lemma 2.4 for every $\gamma \in \Gamma$. This condition is represented by the nonvanishing of a nonconstant polynomial of degree at most $4D_\gamma^3$ for each $\gamma \in \Gamma$.
- An algorithm for the computation of the minimal polynomial of u in \hat{V} to be described below can be extended to a computation of a geometric solution of \hat{V} according to Lemma 2.4. This application of Lemma 2.4 requires that the coefficient vector of the linear form u does not annihilate a nonconstant polynomial of degree at most $4D^4$.

Fix $\rho \geq 2$. From Theorem 2.2 it follows that a linear form u satisfying these conditions can be obtained by randomly choosing its coefficients from the set $\{1, \dots, 6\rho D^4\}$ with error probability at most $1/\rho$.

Next we compute the monic minimal polynomial $\hat{m}_u \in \mathbb{Q}(T)[Y]$ of the linear form u in the curve \hat{V} introduced in (3.2). For this purpose, we approximate the Puiseux series expansions of the branches of \hat{V} lying above 0 by means of a symbolic

(Newton–Hensel) “lifting” of the common zeros of the zero-dimensional varieties $V_{0,\gamma} \subset \mathbb{A}^n$ defined by the polynomials (3.4 _{γ}) for all $\gamma \in \Gamma$ (see Section 4).

This in turn requires the computation of a geometric solution of $V_{0,\gamma}$ for every $\gamma \in \Gamma$. By means of a change of variables we put the system $h_{1,\gamma}^{(0)} = \dots = h_{n,\gamma}^{(0)} = 0$ defining the variety $V_{0,\gamma}$ into a “diagonal” form (see Subsection 4.1 below), which allows us to compute the minimal polynomial $m_{u,\gamma}^{(0)}$ of u in $V_{0,\gamma}$. Since the linear form u satisfies condition 2 of the statement of Lemma 2.4, from this procedure we derive an algorithm computing a geometric solution of $V_{0,\gamma}$ according to Lemma 2.4.

Then we “lift” this geometric solution to a suitable (non-archimedean) approximation \tilde{m}_γ of a factor m_γ (over $\overline{\mathbb{Q}(T)}$) of the desired minimal polynomial \hat{m}_u of u . In the next step we obtain the minimal polynomial $\hat{m}_u = \prod_{\gamma \in \Gamma} m_\gamma$ from the approximate factors \tilde{m}_γ , namely, we compute the dense representation of the coefficients (in $\mathbb{Q}(T)$) of \hat{m}_u , using Padé approximation (see Subsection 4.2 below). Finally, we apply the proof of Lemma 2.4 to derive an algorithm for computing a geometric solution of the variety \hat{V} .

In the last step we compute a geometric solution of the variety V_1 by substituting 1 for T in the polynomials that form the geometric solution of \hat{V} .

The whole algorithm for solving the system $h_1 = \dots = h_n = 0$ may be briefly sketched as follows:

Algorithm 3.5.

- Choose the coefficients of a linear form $u \in \mathbb{Q}[X]$ at random from the set $\{1, \dots, 6\rho D^4\}$.
- For each $\gamma \in \Gamma$:
 - Find a geometric solution of the variety $V_{0,\gamma}$ defined in (3.5).
 - Obtain a straight-line program for the polynomials $h_{1,\gamma}, \dots, h_{n,\gamma}$ defined in (3.6) from the coefficients of h_1, \dots, h_n and the entries of $\gamma \in \mathbb{Z}^{n+1}$.
 - “Lift” the computed geometric solution of $V_{0,\gamma}$ to an approximation \tilde{m}_γ of the factor m_γ of \hat{m}_u by means of a symbolic Newton–Hensel procedure.
- Obtain a geometric solution of the curve \hat{V} :
 - Compute the approximation $\tilde{m}_u := \prod_{\gamma \in \Gamma} \tilde{m}_\gamma$ of \hat{m}_u .
 - Compute the dense representation of \hat{m}_u from \tilde{m}_u using Padé approximation.
 - Find a geometric solution of \hat{V} applying the proof of Lemma 2.4.
- Substitute 1 for T in the polynomials which form the geometric solution of \hat{V} computed in the previous step to obtain a geometric solution of the variety V_1 .

4. SOLUTION OF THE VARIETY \hat{V}

4.1. Geometric solutions of the starting varieties. In this subsection we exhibit an algorithm that computes, for a given inner normal $\gamma \in \Gamma$, a geometric solution of the variety $V_{0,\gamma} \subset (\mathbb{C}^*)^n$ defined by the polynomials $h_{i,\gamma}^{(0)}$ ($1 \leq i \leq n$) for polynomials h_1, \dots, h_n satisfying assumptions (H1) and (H2). This algorithm is based on the procedure presented in [25].

Fix a cell $C = (C^{(1)}, \dots, C^{(s)})$ of type (k_1, \dots, k_s) of the given fine-mixed subdivision of \mathcal{A} and let $\gamma \in \Gamma$ be its associated inner normal. For $1 \leq \ell \leq s$, we denote by $h_1^{(\ell)}, \dots, h_{k_\ell}^{(\ell)}$ the polynomials in the set $\{h_{1,\gamma}^{(0)}, \dots, h_{n,\gamma}^{(0)}\}$ that are supported in $C^{(\ell)}$. In the sequel, whenever there is no risk of confusion we will not write the subscript γ indicating which cell we are considering.

Our hypotheses imply that $h_1^{(\ell)}, \dots, h_{k_\ell}^{(\ell)}$ are \mathbb{Q} -linear combinations of precisely $k_\ell + 1$ monomials in $\mathbb{Q}[X]$ and, up to a multiplication by a monomial, we may assume one of them to be the constant term. Denote these monomials by $X^{\alpha_{\ell,0}}, \dots, X^{\alpha_{\ell,k_\ell}}$, with $\alpha_{\ell,0} := 0 \in \mathbb{Z}^n$. Let $\widetilde{\mathcal{M}}^{(\ell)}$ be the matrix of $\mathbb{Q}^{k_\ell \times (k_\ell + 1)}$ for which the following equality holds in $\mathbb{Q}[X, X^{-1}]^{k_\ell}$:

$$(4.1) \quad \widetilde{\mathcal{M}}^{(\ell)} \begin{pmatrix} X^{\alpha_{\ell,k_\ell}} \\ \vdots \\ X^{\alpha_{\ell,0}} \end{pmatrix} = \begin{pmatrix} h_1^{(\ell)} \\ \vdots \\ h_{k_\ell}^{(\ell)} \end{pmatrix},$$

and let $\mathcal{M}^{(\ell)}$ denote the square $(k_\ell \times k_\ell)$ -matrix obtained by deleting the last column from $\widetilde{\mathcal{M}}^{(\ell)}$. Set

$$\mathcal{M} := \begin{pmatrix} \mathcal{M}^{(1)} & 0 & \dots & 0 \\ 0 & \mathcal{M}^{(2)} & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & \mathcal{M}^{(s)} \end{pmatrix},$$

where 0 here represents different block matrices with all its entries equal to 0 in \mathbb{Q} . Then \mathcal{M} is the matrix defined by the coefficients of the nonconstant terms of the (Laurent) polynomials $h_{1,\gamma}^{(0)}, \dots, h_{n,\gamma}^{(0)}$, up to a translation.

Due to condition (H2) we have that the matrix \mathcal{M} is invertible, which in turn implies that the square matrices $\mathcal{M}^{(\ell)}$ are invertible for $1 \leq \ell \leq s$. Following [25], we apply Gaussian elimination to the matrix $\widetilde{\mathcal{M}}^{(\ell)}$ for $1 \leq \ell \leq s$ and obtain a set of $k_\ell + 1$ binomials

$$\begin{pmatrix} 1 & 0 & 0 & \dots & -c_{\alpha_{\ell,k_\ell}} \\ 0 & 1 & 0 & \dots & -c_{\alpha_{\ell,k_\ell-1}} \\ \vdots & & & \ddots & \\ 0 & 0 & \dots & 1 & -c_{\alpha_{\ell,1}} \end{pmatrix} \begin{pmatrix} X^{\alpha_{\ell,k_\ell}} \\ X^{\alpha_{\ell,k_\ell-1}} \\ \vdots \\ X^{\alpha_{\ell,1}} \end{pmatrix} = \begin{pmatrix} X^{\alpha_{\ell,k_\ell}} - c_{\alpha_{\ell,k_\ell}} \\ X^{\alpha_{\ell,k_\ell-1}} - c_{\alpha_{\ell,k_\ell-1}} \\ \vdots \\ X^{\alpha_{\ell,1}} - c_{\alpha_{\ell,1}} \end{pmatrix}$$

that generate the same linear subspace of $\mathbb{Q}[X, X^{-1}]$ as the polynomials in (4.1). Therefore, for $1 \leq \ell \leq s$ the set of common zeros in $(\mathbb{C}^*)^n$ of the polynomials $h_1^{(\ell)}, \dots, h_{k_\ell}^{(\ell)}$ is given by the system $X^{\alpha_{\ell,k_\ell}} = c_{\alpha_{\ell,k_\ell}}, \dots, X^{\alpha_{\ell,1}} = c_{\alpha_{\ell,1}}$. Putting these s systems together, we obtain a binomial system of the form

$$(4.2) \quad X^{\alpha_1} = p_1, \dots, X^{\alpha_n} = p_n,$$

with $\alpha_i \in \mathbb{Z}^n$ and $p_i \in \mathbb{Q} \setminus \{0\}$ ($1 \leq i \leq n$), that defines the variety $V_{0,\gamma}$. Note that the second part of condition (H2) ensures the non-vanishing of the constants p_i for $1 \leq i \leq n$.

Now, let \mathcal{E} denote the $(n \times n)$ -matrix whose columns are the exponent vectors $\alpha_1, \dots, \alpha_n$. Using [51, Proposition 8.10], we obtain unimodular matrices $K = (k_{i,j})_{1 \leq i,j \leq n}$, $L = (l_{i,j})_{1 \leq i,j \leq n}$ of $\mathbb{Z}^{n \times n}$, and a diagonal matrix $\text{diag}(r_1, \dots, r_n) \in$

$\mathbb{Z}^{n \times n}$ which give the Smith Normal Form for \mathcal{E} , i.e., matrices such that the identity

$$(4.3) \quad K \cdot \mathcal{E} \cdot L = \text{diag}(r_1, \dots, r_n)$$

holds in $\mathbb{Z}^{n \times n}$. We observe that the upper bound

$$(4.4) \quad \log \|K\| \leq (4n + 5)(\log n + \log \|\mathcal{E}\|)$$

holds, where $\|A\|$ denotes the maximum of the absolute value of the entries of a given matrix A [51, Proposition 8.10].

Let Z_1, \dots, Z_n be new indeterminates, and write $Z := (Z_1, \dots, Z_n)$. We introduce the change of coordinates given by $X_i := Z_1^{k_{1,i}} \cdots Z_n^{k_{n,i}}$ for $1 \leq i \leq n$. Making this change of coordinates in (4.2) we obtain the system

$$Z^{K\alpha_1} = p_1, \dots, Z^{K\alpha_n} = p_n,$$

which is equivalent to the “diagonal” system

$$Z_j^{r_j} = \prod_{i=1}^n (Z^{K\alpha_i})^{l_{i,j}} = \prod_{i=1}^n p_i^{l_{i,j}} =: q_j \quad (1 \leq j \leq n).$$

Inverting some of the coefficients q_j if necessary we may assume without loss of generality that the integers r_1, \dots, r_n are positive.

We first describe an algorithm for computing a geometric solution of the variety $V_{0,\gamma} \subset \mathbb{A}^n$ in the coordinate system of \mathbb{A}^n defined by Z_1, \dots, Z_n . This algorithm takes as input the set of polynomials $Z_1^{r_1} - q_1, \dots, Z_n^{r_n} - q_n \in \mathbb{Q}[Z_1, \dots, Z_n]$ defining $V_{0,\gamma}$ in the coordinates Z_1, \dots, Z_n , and outputs a linear form $\tilde{u} \in \mathbb{Q}[Z_1, \dots, Z_n]$ which separates the points of $V_{0,\gamma}$, the minimal polynomial $m_{\tilde{u}} \in \mathbb{Q}[Y]$ of \tilde{u} in $V_{0,\gamma}$ and the parametrizations of Z_1, \dots, Z_n by the zeros of $m_{\tilde{u}}$.

For this purpose, assume that we are given a linear form $\tilde{u} := \tilde{u}_1 Z_1 + \cdots + \tilde{u}_n Z_n \in \mathbb{Q}[Z_1, \dots, Z_n]$ which separates the points of $V_{0,\gamma}$. Observe that the fact that \tilde{u} is a separating linear form for $V_{0,\gamma}$ implies that $\tilde{u}_i \neq 0$ holds for $i = 1, \dots, n$. Let Y, \tilde{Y} be new indeterminates and let $m_1, \dots, m_n \in \mathbb{Q}[Y]$ be the sequence of polynomials defined recursively by:

$$(4.5) \quad m_1 := \tilde{u}_1^{-r_1} Y^{r_1} - q_1, \quad m_i := \text{Res}_{\tilde{Y}}(\tilde{u}_i^{-r_i} (Y - \tilde{Y})^{r_i} - q_i, m_{i-1}(\tilde{Y})) \quad \text{for } 2 \leq i \leq n.$$

We claim that the polynomial m_n equals (up to scaling by a nonzero element of \mathbb{Q}) the minimal polynomial $m_{\tilde{u}} \in \mathbb{Q}[Y]$ of the coordinate function induced by \tilde{u} in the \mathbb{Q} -algebra extension $\mathbb{Q} \hookrightarrow \mathbb{Q}[V_{0,\gamma}]$. Indeed, for every $2 \leq i \leq n$, the polynomial $m_i(Y)$ is a linear combination of $\tilde{u}_i^{-r_i} (Y - \tilde{Y})^{r_i} - q_i$ and $m_{i-1}(\tilde{Y})$ over $\mathbb{Q}[Y, \tilde{Y}]$. Let $u^{(i)} := \tilde{u}_1 Z_1 + \cdots + \tilde{u}_i Z_i$ for $1 \leq i \leq n$. Then, the identity $\tilde{u}_i^{-r_i} (u^{(i)} - u^{(i-1)})^{r_i} - q_i = 0$ holds in $\mathbb{Q}[V_{0,\gamma}]$. Thus, assuming inductively that $m_{i-1}(u^{(i-1)}) = 0$ in $\mathbb{Q}[V_{0,\gamma}]$, it follows that $m_i(u^{(i)}) = 0$ in $\mathbb{Q}[V_{0,\gamma}]$ as well. Taking into account that $\deg m_n \leq r_1 \cdots r_n$ and that $m_{\tilde{u}}$ is a nonzero polynomial of degree $D_\gamma := r_1 \cdots r_n = \#(V_{0,\gamma})$, we conclude that our claim holds.

In order to compute the polynomial $m_{\tilde{u}}$, we compute the resultants in (4.5). Since the resultant $\text{Res}_{\tilde{Y}}(\tilde{u}_i^{-r_i} (Y - \tilde{Y})^{r_i} - q_i, m_{i-1}(\tilde{Y}))$ is a polynomial of $\mathbb{Q}[Y]$ of degree $r_1 \cdots r_i$, by univariate interpolation in the variable \tilde{Y} we reduce its computation to the computation of $r_1 \cdots r_i + 1$ resultants of univariate polynomials in $\mathbb{Q}[\tilde{Y}]$. This interpolation step requires $O(M(r_1^2 \cdots r_i^2))$ arithmetic operations in \mathbb{Q} and does not require any division by a nonconstant polynomial in the coefficients $\tilde{u}_1, \dots, \tilde{u}_n$ (see,

e.g., [9], [10]). Each univariate resultant can be computed using the algorithms in e.g. [5], [56] with $M(r_1 \cdots r_i)$ arithmetic operations in \mathbb{Q} . Altogether, we obtain an algorithm for computing the minimal polynomial $m_{\tilde{u}}$ which performs $O(M(D_\gamma^2))$ arithmetic operations in \mathbb{Q} .

Next, we extend this algorithm to an algorithm for computing a geometric solution of $V_{0,\gamma}$ as explained in Subsection 2.3. We obtain the following result:

Proposition 4.1. *Suppose that the coefficients of the linear form \tilde{u} are randomly chosen in the set $\{1, \dots, 4n\rho D_\gamma^3\}$, where ρ is a fixed positive integer. Then the algorithm described above computes a geometric solution of the variety $V_{0,\gamma}$ (in the coordinate system Z_1, \dots, Z_n) with error probability at most $1/\rho$ using $O(nM(D_\gamma^2))$ arithmetic operations in \mathbb{Q} .*

Proof. As proved by our previous arguments, it is clear that the algorithm described computes a geometric solution of $V_{0,\gamma}$ with the stated number of arithmetic operations in \mathbb{Q} . There remains to analyze its error probability.

The only probabilistic step of the algorithm is the choice of the coefficients of the linear form \tilde{u} , which must satisfy two requirements. First, \tilde{u} must separate the points of the variety $V_{0,\gamma}$. Since $V_{0,\gamma}$ consists of D_γ distinct points of \mathbb{A}^n , from Theorem 2.2 it follows that for a random choice of the coefficients of \tilde{u} in the set $\{1, \dots, 4n\rho D_\gamma^3\}$, the linear form \tilde{u} separates the points of $V_{0,\gamma}$ with error probability at most $1/4n\rho D_\gamma \leq 1/2\rho$.

The second requirement concerns the computation of the univariate resultants of the generic versions of the polynomials in (4.5). This is required in order to extend the algorithm for computing the minimal polynomial $m_{\tilde{u}}$ to an algorithm for computing a geometric solution of the variety $V_{0,\gamma}$. We use a fast algorithm for computing resultants over $\mathbb{Q}(\Lambda)$ based on the Extended Euclidean Algorithm (EEA for short). We shall perform the EEA over the ring of power series $\mathbb{Q}[[\Lambda - \tilde{u}]]$, truncating all the intermediate results up to order 2. Therefore, the choice of the coefficients of \tilde{u} must guarantee that all the elements of $\mathbb{Q}[\Lambda]$ which have to be inverted during the execution of the EEA are invertible elements of the ring $\mathbb{Q}[[\Lambda - \tilde{u}]]$.

For this purpose, we observe that, similarly to the proof of [56, Theorem 6.52], one deduces that all the denominators of the elements of $\mathbb{Q}(\Lambda)$ arising during the application of the EEA to the generic version of the polynomials $\tilde{u}_i^{-r_i}(\alpha - u^{(i-1)})^{r_i} - q_i$ and $m_{i-1}(u^{(i-1)})$ are divisors of at most $r_1 \cdots r_{i-1}$ polynomials of $\mathbb{Q}[\Lambda]$ of degree $2r_1 \cdots r_i$ for any $\alpha \in \mathbb{Q}$. This EEA step must be executed for $r_1 \cdots r_i$ distinct values of $\alpha \in \mathbb{Q}$, in order to perform the interpolation step. Hence the product of the denominators arising during all the applications of the EEA has degree at most $2nD_\gamma^3$. Therefore, from Theorem 2.2 we conclude that for a random choice of its coefficients in the set $\{1, \dots, 4n\rho D_\gamma^3\}$, the linear form \tilde{u} satisfies our second requirement with error probability at most $1/2\rho$.

The lemma follows putting both error probability estimates together. \square

Finally, we compute a geometric solution of the variety $V_{0,\gamma}$ in the original coordinate system defined by X_1, \dots, X_n .

For this purpose, we compute the minimal polynomial $m_u \in \mathbb{Q}[Y]$ of a linear form $u = u_1X_1 + \cdots + u_nX_n \in \mathbb{Q}[X_1, \dots, X_n]$ in $V_{0,\gamma}$. Let $V_{0,\gamma} := \{x_0^{(1,\gamma)}, \dots, x_0^{(D_\gamma,\gamma)}\}$. Then we have $m_u(Y) = \prod_{j=1}^{D_\gamma} (Y - u(x_0^{(j,\gamma)}))$. In order to compute m_u , we use

the polynomials $m_{\tilde{u}}, \tilde{w}_1, \dots, \tilde{w}_n$ which form the previously computed geometric solution of $V_{0,\gamma}$ in the variables Z_1, \dots, Z_n : from the identities $X_i := Z_1^{k_{1,i}^{(\gamma)}} \dots Z_n^{k_{n,i}^{(\gamma)}}$ ($1 \leq i \leq n$) we deduce that m_u equals the minimal polynomial of the image of the projection $\eta_u : V_{0,\gamma} \rightarrow \mathbb{A}^1$ defined by $\eta_u(z_1, \dots, z_n) := \sum_{i=1}^n u_i z_1^{k_{1,i}^{(\gamma)}} \dots z_n^{k_{n,i}^{(\gamma)}}$. Now, the identities $Z_i = \tilde{w}_i(\tilde{u})$, which hold in $\mathbb{Q}[V_{0,\gamma}]$ for $1 \leq i \leq n$, imply that

$$(4.6) \quad u = \sum_{i=1}^n u_i (\tilde{w}_1(\tilde{u}))^{k_{1,i}^{(\gamma)}} \dots (\tilde{w}_n(\tilde{u}))^{k_{n,i}^{(\gamma)}}$$

holds in $\mathbb{Q}[V_{0,\gamma}]$, from which we easily conclude that m_u satisfies the following identity:

$$(4.7) \quad m_u(Y) = \text{Res}_{\tilde{Y}} \left(Y - \sum_{i=1}^n u_i (\tilde{w}_1(\tilde{Y}))^{k_{1,i}^{(\gamma)}} \dots (\tilde{w}_n(\tilde{Y}))^{k_{n,i}^{(\gamma)}}, m_{\tilde{u}}(\tilde{Y}) \right).$$

We compute the monomials $(\tilde{w}_1(\tilde{u}))^{k_{1,i}^{(\gamma)}} \dots (\tilde{w}_n(\tilde{u}))^{k_{n,i}^{(\gamma)}}$ ($1 \leq i \leq n$) in the right-hand side of (4.6) modulo $m_{\tilde{u}}(Y)$, with $O(n^2 \log(\max_{i,j} |k_{i,j}^{(\gamma)}|) M(D_\gamma))$ additional arithmetic operations in \mathbb{Q} . From (4.4) it follows that

$$O(n^2 \log(\max_{i,j} |k_{i,j}^{(\gamma)}|) M(D_\gamma)) = O(n^3 \log(n \|\mathcal{E}_\gamma\|) M(D_\gamma)),$$

where \mathcal{E}_γ is the matrix of the exponents of the cell corresponding to the inner normal γ . Observe that all these steps are independent of the coefficients of the linear form u we are considering and therefore do not introduce any division by a nonconstant polynomial in the coefficients u_1, \dots, u_n .

In the next step we compute the right-hand side of (4.6) modulo $m_{\tilde{u}}(Y)$, with $O(nD_\gamma)$ arithmetic operations in \mathbb{Q} . Then we compute the resultant (4.7) by a process which interpolates (4.7) in the variable Y to reduce the question to the computation of $D_\gamma + 1$ univariate resultants, in the same way as for the computation of the resultants in (4.5). This requires $O(M(D_\gamma)^2)$ arithmetic operations in \mathbb{Q} .

If the linear form u separates the points of $V_{0,\gamma}$, then we can extend the algorithm for computing $m_u(Y)$ to an algorithm for computing a geometric solution of $V_{0,\gamma}$ with the algorithm underlying the proof of Lemma 2.4. This extension requires that the coefficients u_1, \dots, u_n of the linear form u do not annihilate the denominators in $\mathbb{Q}[\Lambda]$ which arise from the application of the algorithm described above to the generic version $\Lambda_1 X_1 + \dots + \Lambda_n X_n$ of the linear form u . Such denominators arise only during the computation of the generic version of the resultant (4.7). Hence, with a similar analysis as in the proof of Proposition 4.1, we conclude that, if the coefficients of u are chosen randomly in the set $\{1, \dots, 4\rho D_\gamma^3\}$, then the error probability of our algorithm is bounded by $1/\rho$. In conclusion, we have:

Proposition 4.2. *Suppose that we are given a geometric solution of $V_{0,\gamma}$ in the coordinate system Z_1, \dots, Z_n , as provided by the algorithm underlying Proposition 4.1, and the coefficients of the linear form u are randomly chosen in the set $\{1, \dots, 4\rho D_\gamma^3\}$, where ρ is a fixed positive integer. Then the algorithm described above computes a geometric solution of the variety $V_{0,\gamma}$ with error probability at most $1/\rho$ using $O(n^3 \log(n \|\mathcal{E}_\gamma\|) M(D_\gamma)^2)$ arithmetic operations in \mathbb{Q} .*

Finally, from Propositions 4.1 and 4.2 and the fact that $\|\mathcal{E}_\gamma\|$ is bounded by $\mathcal{Q} := 2 \max_{1 \leq i \leq n} \{\|q\|; q \in \Delta_i\}$, we immediately deduce the following result:

Theorem 4.3. *Suppose that the coefficients of the linear forms \tilde{u} and u of the statement of Propositions 4.1 and 4.2 are chosen at random in the set $\{1, \dots, 4n\rho D^3\}$, where ρ is a fixed positive integer. Then the algorithm underlying Propositions 4.1 and 4.2 computes a geometric solution of the varieties $V_{0,\gamma}$ for all $\gamma \in \Gamma$ with error probability at most $2/\rho$ using $O(n^3 \log(nQ)M(D)^2)$ arithmetic operations in \mathbb{Q} .*

4.2. The computation of a geometric solution of the first deformation.

The second step of our algorithm consists in the computation of a geometric solution of the curve \widehat{V} of (3.2). This will be done by “lifting” the geometric solutions of the varieties $V_{0,\gamma}$ computed in the previous section for all $\gamma \in \Gamma$.

We recall the definition of the variety \widehat{V} . Let I denote the ideal of $\mathbb{Q}[X, T]$ generated by the polynomials $\widehat{h}_1, \dots, \widehat{h}_n$ of (3.1), which form the polyhedral deformation of the generic polynomials h_1, \dots, h_n , and let J denote the Jacobian determinant of $\widehat{h}_1, \dots, \widehat{h}_n$ with respect to the variables X_1, \dots, X_n . Let $V(I)$ be the set of common zeros in \mathbb{A}^{n+1} of $\widehat{h}_1, \dots, \widehat{h}_n$. Then $\widehat{V} := V(I : J^\infty)$.

Alternatively, let $\pi : V(I) \rightarrow \mathbb{A}^1$ be the linear projection defined by $\pi(x, t) = t$. Consider the decomposition of $V(I)$ into its irreducible components $V(I) = \bigcup_{i=1}^{r+s} \mathcal{C}_i$. Suppose that the restriction $\pi|_{\mathcal{C}_i} : \mathcal{C}_i \rightarrow \mathbb{A}^1$ of the projection π is dominant for $1 \leq i \leq r$ and is not dominant for $r+1 \leq i \leq s$. We shall show that $\widehat{V} := \bigcup_{i=1}^r \mathcal{C}_i$ holds, i.e., \widehat{V} is the union of all the irreducible components of $V(I)$ which project dominantly over \mathbb{A}^1 . Furthermore, we shall show that $\widehat{V} \subset \mathbb{A}^{n+1}$ is a curve which constitutes a suitable deformation of the variety defined by the system $h_1 = \dots = h_n = 0$. For this purpose, we shall use the following technical lemma:

Lemma 4.4. *Let $F_1, \dots, F_n \in \mathbb{Q}[X, T]$ and $\mathcal{V} := \{(x, t) \in \mathbb{A}^{n+1} : F_1(x, t) = 0, \dots, F_n(x, t) = 0\}$. Set $I := (F_1, \dots, F_n) \subset \mathbb{Q}[X, T]$ and let J denote the Jacobian determinant of F_1, \dots, F_n with respect to the variables X . Consider the linear projection $\pi : \mathcal{V} \rightarrow \mathbb{A}^1$ defined by $\pi(x, t) := t$. Assume that $\#\pi^{-1}(t) \leq D$ holds for generic values of $t \in \mathbb{A}^1$ and that there exists a point $t_0 \in \mathbb{A}^1$ such that the fiber $\pi^{-1}(t_0)$ is a zero-dimensional variety of degree D with $J(x, t_0) \neq 0$ for every $(x, t_0) \in \pi^{-1}(t_0)$.*

Let \mathcal{V}_{dom} be the union of all the irreducible components \mathcal{C} of \mathcal{V} with $\overline{\pi(\mathcal{C})} = \mathbb{A}^1$. Then:

- \mathcal{V}_{dom} is a nonempty equidimensional variety of dimension 1.
- \mathcal{V}_{dom} is the union of all the irreducible components of \mathcal{V} having a non-empty intersection with $\pi^{-1}(t_0)$.
- $\mathcal{V}_{\text{dom}} = V(I : J^\infty)$.
- The restriction $\pi|_{\mathcal{V}_{\text{dom}}} : \mathcal{V}_{\text{dom}} \rightarrow \mathbb{A}^1$ is a dominant map of degree D .

Proof. First we observe that $\dim(\mathcal{C}) \geq 1$ for each irreducible component \mathcal{C} of \mathcal{V} , since \mathcal{V} is defined by n polynomials in an $(n+1)$ -dimensional space.

Let \mathcal{C} be an irreducible component of \mathcal{V} for which $\pi^{-1}(t_0) \cap \mathcal{C} \neq \emptyset$ holds. Consider the restriction $\pi|_{\mathcal{C}} : \mathcal{C} \rightarrow \mathbb{A}^1$ of the projection map π . Then we have that $\pi|_{\mathcal{C}}^{-1}(t_0)$ is a nonempty zero-dimensional variety, which implies that the generic fiber of $\pi|_{\mathcal{C}}$ is either zero-dimensional or empty. Since $\dim(\mathcal{C}) \geq 1$, the Theorem on the Dimension of Fibers implies that $\dim(\mathcal{C}) = 1$ and that $\pi|_{\mathcal{C}} : \mathcal{C} \rightarrow \mathbb{A}^1$ is a dominant map with generically-finite fibers. This shows that $\mathcal{C} \subset \mathcal{V}_{\text{dom}}$ and, in particular, that \mathcal{V}_{dom} is nonempty.

Conversely, we have that $\pi^{-1}(t_0) \cap \mathcal{C} \neq \emptyset$ holds for any irreducible component \mathcal{C} of \mathcal{V}_{dom} . Indeed, assume on the contrary the existence of an irreducible component \mathcal{C}_0 not satisfying this condition. Then, there is a point $t_1 \in \mathbb{A}^1$ having a finite fiber $\pi^{-1}(t_1)$ such that $\pi|_{\mathcal{C}_0}^{-1}(t_1)$ and $\pi|_{\mathcal{C}}^{-1}(t_1)$ have maximal cardinality for every \mathcal{C} with $\mathcal{C} \cap \pi^{-1}(t_0) \neq \emptyset$. This implies that $\#\pi^{-1}(t_1) > \#\pi^{-1}(t_0) = D$, leading to a contradiction.

We conclude that \mathcal{V}_{dom} is the nonempty equidimensional variety of dimension 1 which consists of all the irreducible components \mathcal{C} of \mathcal{V} with $\pi^{-1}(t_0) \cap \mathcal{C} \neq \emptyset$. Furthermore, this shows that the restriction $\pi|_{\mathcal{V}_{\text{dom}}} : \mathcal{V}_{\text{dom}} \rightarrow \mathbb{A}^1$ is a dominant map of degree D .

Finally we show that the identity $\mathcal{V}_{\text{dom}} = V(I : J^\infty)$ holds. First, note that the irreducible components of $V(I : J^\infty)$ are all the irreducible components of \mathcal{V} where the Jacobian J does not vanish identically. Thus, it is clear that $\mathcal{V}_{\text{dom}} \subset V(I : J^\infty)$, since J does not vanish at the points of $\pi^{-1}(t_0) \cap \mathcal{C}$ for each irreducible component \mathcal{C} of \mathcal{V}_{dom} . On the other hand, if \mathcal{C} is an irreducible component of \mathcal{V} for which the projection $\pi|_{\mathcal{C}} : \mathcal{C} \rightarrow \mathbb{A}^1$ is not dominant, then \mathcal{C} is the set of common zeros of the polynomials $F_1, \dots, F_n, T - t_{\mathcal{C}}$ for some value $t_{\mathcal{C}}$. Since $\dim(\mathcal{C}) \geq 1$, we have that the Jacobian matrix $\partial(F_1, \dots, F_n, T - t_{\mathcal{C}})/\partial(X_1, \dots, X_n, T)$ is singular at every point $(x, t_{\mathcal{C}})$ of \mathcal{C} . Hence, its determinant, which equals J , vanishes over \mathcal{C} . \square

Now we return to the study of the variety \widehat{V} and show that the assumptions of Lemma 4.4 hold. Observe that $\pi^{-1}(t) = V_t \times \{t\}$ holds for every $t \in \mathbb{A}^1$, where $V_t := \{x \in \mathbb{A}^n : \widehat{h}_1(x, t) = 0, \dots, \widehat{h}_n(x, t) = 0\}$. Furthermore, the polynomials $\widehat{h}_1(X, t), \dots, \widehat{h}_n(X, t)$ are obtained by a suitable substitution of the variables Ω of the generic polynomials $H_1, \dots, H_n \in \mathbb{Q}[\Omega, X]$ with supports $\Delta_1, \dots, \Delta_n$ introduced in (3.7). Indeed, if $c = (c_1, \dots, c_n)$ is the vector of coefficients of h_1, \dots, h_n , the coefficient vector of $\widehat{h}_i(X, t)$ ($1 \leq i \leq n$) is $(c_{i,q} t^{\omega_i(q)})_{q \in \Delta_i}$ for every $t \in \mathbb{A}^1$. By Lemma 3.1, there exists a nonzero polynomial $P^{(0)} \in \mathbb{Q}[\Omega]$ such that for any $c' = (c'_1, \dots, c'_n)$ with $P^{(0)}(c') \neq 0$ the associated sparse system defines a zero-dimensional variety. In particular, the coefficients $c = (c_1, \dots, c_n)$ of our input polynomials $h_1 := H_1(c_1, X), \dots, h_n := H_n(c_n, X)$ satisfy $P^{(0)}(c) \neq 0$. This shows that the polynomial $P_T^{(0)} \in \mathbb{Q}[T]$ obtained by substituting $\Omega_{i,q} \mapsto c_{i,q} T^{\omega_i(q)}$ ($1 \leq i \leq n, q \in \Delta_i$) in the polynomial $P^{(0)}$ is nonzero, since it does not vanish at $T = 1$. We conclude that V_t is a zero-dimensional variety for all but a finite number of $t \in \mathbb{A}^1$. Thus, $\pi^{-1}(t)$ is finite for generic values of $t \in \mathbb{A}^1$.

Finally, by condition (H1), the fiber $\pi^{-1}(1) = V(h_1, \dots, h_n) \times \{1\}$ is a zero-dimensional variety of degree $D = \deg(\pi)$ and the Jacobian determinant $J := \det(\partial \widehat{h}_i / \partial X_j)_{1 \leq i, j \leq n}$ does not vanish at any of its points. On the other hand, the fact that $\#\pi^{-1}(t) \leq D$ holds for generic values $t \in \mathbb{A}^1$ follows from the BKK theorem.

This shows that the variety $V(I)$ and its defining polynomials $\widehat{h}_1, \dots, \widehat{h}_n$ satisfy all the assumptions of Lemma 4.4. Thus, we have:

Lemma 4.5. *The variety $\widehat{V} \subset \mathbb{A}^{n+1}$ is a curve. Furthermore, every irreducible component of \widehat{V} has a nonempty intersection with the fiber $\pi^{-1}(1)$ of the projection map $\pi : \widehat{V} \rightarrow \mathbb{A}^1$.*

4.2.1. *Generic linear projections of \widehat{V} .* In order to compute a geometric solution of the space curve \widehat{V} , we shall first exhibit a procedure for computing the minimal polynomial of a generic linear projection of \widehat{V} . Let $u \in \mathbb{Q}[X_1, \dots, X_n]$ be a linear form which separates the points of the “initial varieties” $V_{0,\gamma}$ for all the inner normals $\gamma := (\gamma_1, \dots, \gamma_{n+1})$ of the lower facets of the polyhedral deformation under consideration. Let $\pi_u : \widehat{V} \rightarrow \mathbb{A}^2$ be the morphism defined by $\pi_u(x, t) := (t, u(x))$. Since the projection map $\pi : \widehat{V} \rightarrow \mathbb{A}^1$ defined by $\pi(x, t) := t$ is dominant, it follows that the Zariski closure of the image of π_u is a \mathbb{Q} -definable hypersurface of \mathbb{A}^2 . Denote by $M_u \in \mathbb{Q}[T, Y]$ a minimal defining polynomial for this hypersurface. For the sake of the argument, we shall assume further that the identity $\deg(\pi) = D$, and thus $\deg_Y M_u = D$, hold.

We can apply estimate (2.2) of Lemma 2.3 in order to estimate $\deg_T M_u$ in combinatorial terms (compare with [43, Theorem 1.1]). Indeed, let $\widehat{Q}_1, \dots, \widehat{Q}_n \subset \mathbb{R}^{n+1}$ be the Newton polytopes of the polynomials $\widehat{h}_1, \dots, \widehat{h}_n$ of (3.1), and let $\Delta \subset \mathbb{R}^{n+1}$ be the standard unitary simplex in the plane $\{T = 0\}$. Then the following estimate holds:

$$(4.8) \quad \deg_T M_u \leq E := MV_{n+1}(\Delta, \widehat{Q}_1, \dots, \widehat{Q}_n).$$

Furthermore, equality holds in (4.8) for a generic choice of the coefficients of the polynomials \widehat{h}_i and the linear form u .

More precisely, we shall exhibit a procedure for computing the unique monic multiple in $\mathbb{Q}(T)[Y]$ of M_u of degree D . This polynomial can be alternatively defined as explained in what follows:

Since the projection map $\pi : \widehat{V} \rightarrow \mathbb{A}^1$ is dominant, it induces an extension $\mathbb{Q}[T] \hookrightarrow \mathbb{Q}[\widehat{V}]$, where $\mathbb{Q}[\widehat{V}]$ denotes the coordinate ring of \widehat{V} . This variety being a curve, $\mathbb{Q}[\widehat{V}]$ turns out to be a finitely generated $\mathbb{Q}[T]$ -module. Thus, tensoring with $\mathbb{Q}(T)$, we deduce that $\mathbb{Q}[\widehat{V}] \otimes \mathbb{Q}(T)$ is a $\mathbb{Q}(T)$ -vector space of finite dimension. We claim that $\mathbb{Q}[\widehat{V}] \otimes \mathbb{Q}(T) = \mathbb{Q}[V(I)] \otimes \mathbb{Q}(T)$ holds. Indeed, since \widehat{V} consists of the irreducible components of $V(I)$ which are mapped dominantly onto \mathbb{A}^1 by the projection π , for each of the remaining irreducible components \mathcal{C} of $V(I)$, the set $\pi(\mathcal{C}) \subset \mathbb{C}$ is a zero-dimensional \mathbb{Q} -definable variety. This implies that $I(\mathcal{C}) \cap \mathbb{Q}[T] \neq \{0\}$ holds.

Let \widehat{m}_u be the minimal polynomial of u in the extension $\mathbb{Q}(T) \hookrightarrow \mathbb{Q}[\widehat{V}] \otimes \mathbb{Q}(T)$. The fact that $\mathbb{Q}[\widehat{V}] \otimes \mathbb{Q}(T)$ is finite-dimensional $\mathbb{Q}(T)$ -vector space shows that the affine variety $\mathbb{V} := \{\bar{x} \in \mathbb{A}^n(\overline{\mathbb{Q}(T)^*}) : \widehat{h}_1(\bar{x}) = 0, \dots, \widehat{h}_n(\bar{x}) = 0\}$ has dimension zero. Here $\overline{\mathbb{Q}(T)^*} := \bigcup_{q \in \mathbb{N}} \overline{\mathbb{Q}}((T^{1/q}))$ denotes the field of Puiseux series in the variable T over $\overline{\mathbb{Q}}$ (see, e.g., [57]) and $\widehat{h}_1, \dots, \widehat{h}_n$ are considered as elements of $\mathbb{Q}(T)[X]$. Our hypotheses imply that there exist D distinct n -tuples $x^{(\ell)} := (x_1^{(\ell)}, \dots, x_n^{(\ell)}) \in (\overline{\mathbb{Q}(T)^*})^n$ of Puiseux series such that the following equalities hold in $\overline{\mathbb{Q}(T)^*}$ for $1 \leq \ell \leq D$:

$$(4.9) \quad \widehat{h}_1(x^{(\ell)}, T) = 0, \dots, \widehat{h}_n(x^{(\ell)}, T) = 0$$

(see [25]). Since $\mathbb{Q}[\widehat{V}] \otimes \mathbb{Q}(T)$ is the coordinate ring of the $\mathbb{Q}(T)$ -variety \mathbb{V} , from (4.9) we deduce that the dimension of $\mathbb{Q}[\widehat{V}] \otimes \mathbb{Q}(T)$ over $\mathbb{Q}(T)$ equals D . Moreover,

since as a consequence of our assumptions $\deg_Y \hat{m}_u = D$ holds, we conclude that

$$(4.10) \quad \hat{m}_u = \prod_{\ell=1}^D (Y - u(x^{(\ell)})).$$

Since $M_u(T, u(X)) \in I(\hat{V})$, it follows that $M_u(T, u(X)) = 0$ holds in $\mathbb{Q}[\hat{V}] \otimes \mathbb{Q}(T)$, from which we conclude that M_u is a multiple of \hat{m}_u by a factor in $\mathbb{Q}(T)[Y]$. Taking into account that both are polynomials of degree D in the variable Y and that \hat{m}_u is monic in this variable, we deduce that \hat{m}_u is the quotient of M_u by its leading coefficient.

4.2.2. A procedure for computing \hat{m}_u . Now we exhibit a procedure for computing the minimal polynomial \hat{m}_u , which is based on the expression (4.10) of \hat{m}_u in terms of the Puiseux expansions (3.3). Then we will apply Lemma 2.4 to this procedure in order to obtain an algorithm for computing a geometric solution of the curve \hat{V} .

With notations as in Section 3.1, let $\Gamma \subset \mathbb{Z}^{n+1}$ be the set of primitive integer vectors of the form $\gamma := (\gamma_1, \dots, \gamma_n, \gamma_{n+1}) \in \mathbb{Z}^{n+1}$ with $\gamma_{n+1} > 0$ for which there is a cell $C = (C^{(1)}, \dots, C^{(s)})$ of type (k_1, \dots, k_s) of the subdivision of \mathcal{A} induced by ω such that \hat{C} has inner normal γ . As asserted in Section 3.1, if $\gamma \in \Gamma$ is the inner normal of the lifting \hat{C} of a cell C of type (k_1, \dots, k_s) , there exist $D_\gamma := k_1! \cdots k_s! \cdot \text{vol}(C)$ vectors of Puiseux series $x^{(j,\gamma)} := (x_1^{(j,\gamma)}, \dots, x_n^{(j,\gamma)}) \in \mathbb{A}^n(\mathbb{Q}(T)^*)$ ($1 \leq j \leq D_\gamma$) of the form

$$x_i^{(j,\gamma)} := \sum_{m \geq 0} x_{i,m}^{(j,\gamma)} T^{\frac{\gamma_i + m}{\gamma_{n+1}}}$$

satisfying (4.9). Considering the projection of the branches of \hat{V} parametrized by the D_γ vectors of Puiseux series $x^{(j,\gamma)}$ for each $\gamma \in \Gamma$, we obtain the following element m_γ of $\mathbb{Q}((T^{1/\gamma_{n+1}}))[Y]$:

$$(4.11) \quad m_\gamma := \prod_{j=1}^{D_\gamma} (Y - u(x^{(j,\gamma)})).$$

From (2.1) we conclude that (4.10) may be expressed in the following way:

$$(4.12) \quad \hat{m}_u = \prod_{\gamma \in \Gamma} m_\gamma.$$

Since \hat{m}_u belongs to $\mathbb{Q}(T)[Y]$ and its primitive multiple $M_u \in \mathbb{Q}[T, Y]$ satisfies the degree estimate $\deg_T M_u \leq E$, in order to compute the dense representation of \hat{m}_u we shall compute the Puiseux expansions of the coefficients of the factors $m_\gamma \in \mathbb{Q}((T^{1/\gamma_{n+1}}))[Y]$ of \hat{m}_u truncated up to order $2E$. Using Padé approximation it is possible to recover the dense representation of \hat{m}_u from this data.

Fix $\gamma \in \Gamma$ and set $\mathbf{x}_m^{(j,\gamma)} := (x_{1,m}^{(j,\gamma)}, \dots, x_{n,m}^{(j,\gamma)})$ for every $m \geq 0$ and $1 \leq j \leq D_\gamma$. Since

$$\hat{h}_i \left(\sum_{m \geq 0} x_{1,m}^{(j,\gamma)} T^{\frac{\gamma_1 + m}{\gamma_{n+1}}}, \dots, \sum_{m \geq 0} x_{n,m}^{(j,\gamma)} T^{\frac{\gamma_n + m}{\gamma_{n+1}}}, T \right) = 0$$

holds for $1 \leq j \leq D_\gamma$ and $1 \leq i \leq n$, we have

$$\begin{aligned}
 0 &= T^{-m_i} \widehat{h}_i \left(\sum_{m \geq 0} x_{1,m}^{(j,\gamma)} T^{\gamma_1+m}, \dots, \sum_{m \geq 0} x_{n,m}^{(j,\gamma)} T^{\gamma_n+m}, T^{\gamma_{n+1}} \right) \\
 (4.13) \quad &= T^{-m_i} \widehat{h}_i \left(T^{\gamma_1} \sum_{m \geq 0} x_{1,m}^{(j,\gamma)} T^m, \dots, T^{\gamma_n} \sum_{m \geq 0} x_{n,m}^{(j,\gamma)} T^m, T^{\gamma_{n+1}} \right) \\
 &= h_{i,\gamma} \left(\sum_{m \geq 0} \mathbf{x}_m^{(j,\gamma)} T^m, T \right),
 \end{aligned}$$

according to (3.6). Therefore the polynomial $m_\gamma(T^{\gamma_{n+1}}, Y) \in \mathbb{Q}((T))[Y]$ can be expressed in terms of the power series solutions $\sigma^{(j,\gamma)} := (\sigma_1^{(j,\gamma)}, \dots, \sigma_n^{(j,\gamma)}) := \sum_{m \geq 0} \mathbf{x}_m^{(j,\gamma)} T^m$ ($1 \leq j \leq D_\gamma$) of $h_{1,\gamma}, \dots, h_{n,\gamma}$. Indeed, from (4.11) it follows that

$$\begin{aligned}
 m_\gamma(T^{\gamma_{n+1}}, Y) &= \prod_{j=1}^{D_\gamma} \left(Y - \sum_{i=1}^n u_i \sum_{m \geq 0} x_{i,m}^{(j,\gamma)} T^{\gamma_i+m} \right) \\
 &= \prod_{j=1}^{D_\gamma} \left(Y - \sum_{m \geq 0} \sum_{i=1}^n u_i x_{i,m}^{(j,\gamma)} T^{\gamma_i+m} \right) \\
 (4.14) \quad &= \prod_{j=1}^{D_\gamma} \left(Y - \sum_{m \geq 0} u_\gamma(\mathbf{x}_m^{(j,\gamma)}) T^m \right) \\
 &= \prod_{j=1}^{D_\gamma} \left(Y - u_\gamma \left(\sum_{m \geq 0} \mathbf{x}_m^{(j,\gamma)} T^m \right) \right) =: m_{u_\gamma}(T, Y),
 \end{aligned}$$

where $u_\gamma := \sum_{i=1}^n u_i T^{\gamma_i} X_i$. We conclude that the Laurent polynomial $m_\gamma(T^{\gamma_{n+1}}, Y) \in \mathbb{Q}((T))[Y]$ may be considered as the minimal polynomial $m_{u_\gamma}(T, Y)$ of the projection induced by u_γ on the subvariety V_γ of $\mathbb{A}^n(\overline{\mathbb{Q}}(T)^*)$ consisting of the set of power series $\{\sigma^{(1,\gamma)}, \dots, \sigma^{(D_\gamma,\gamma)}\}$. This remark will allow us to compute a suitable approximation to the Laurent polynomial $m_\gamma(T^{\gamma_{n+1}}, Y)$ in $\mathbb{Q}((T))[Y]$.

In order to describe this approximation, we introduce the following terminology: for $G, \tilde{G} \in \overline{\mathbb{Q}}((T))$ and $s \in \mathbb{Z}$, we say that \tilde{G} **approximates** G with **precision** s in $\overline{\mathbb{Q}}((T))$ if the Laurent series $G - \tilde{G}$ has order at least $s+1$ in T . We shall use the notation $G \equiv \tilde{G} \pmod{(T^{s+1})}$. Furthermore, if G, \tilde{G} are two elements of a polynomial ring $\overline{\mathbb{Q}}((T))[Y]$, we say that \tilde{G} **approximates** G with **precision** s if every coefficient $\tilde{a} \in \overline{\mathbb{Q}}((T))$ of \tilde{G} approximates the corresponding coefficient $a \in \overline{\mathbb{Q}}((T))$ of G with precision s (in the sense of the previous definition).

Proposition 4.6. *Fix $\gamma := (\gamma_1, \dots, \gamma_n) \in \Gamma$ and assume that a geometric solution of the variety $V_{0,\gamma}$ is given, as provided by Theorem 4.3. Assume further that the coefficients of the linear form u of the given geometric solution of $V_{0,\gamma}$ are randomly chosen in the set $\{1, \dots, 4\rho D_\gamma^3\}$ for a given $\rho \in \mathbb{N}$. Then there is an algorithm which computes an approximation to the polynomial $m_{u_\gamma} \in \mathbb{Q}((T))[Y]$ with precision $2E\gamma_{n+1}$. The procedure requires $O((nL_\gamma + n^\Omega)\mathbf{M}(D_\gamma)(\mathbf{M}(M_\gamma)\mathbf{M}(D_\gamma) + E\gamma_{n+1}))$ arithmetic operations in \mathbb{Q} , where $M_\gamma := \max\{\gamma_1, \dots, \gamma_n\}$ and L_γ is the number of arithmetic operations required to evaluate the polynomials $h_{i,\gamma}$ of (3.6), and has error probability at most $2/\rho$.*

Proof. Let notations and assumptions be as before. In order to compute the required approximation of the polynomial m_{u_γ} we first compute the corresponding approximation of the polynomials that form a geometric solution of the variety $V_\gamma := \{\sigma^{(j,\gamma)}; 1 \leq j \leq D_\gamma\}$. Observe that

$$\begin{aligned}
 \{\sigma^{(j,\gamma)}(0); 1 \leq j \leq D_\gamma\} &= \{\mathbf{x}_0^{(j,\gamma)}; 1 \leq j \leq D_\gamma\} \\
 &= V(h_{1,\gamma}^{(0)}, \dots, h_{n,\gamma}^{(0)}) \cap (\mathbb{C}^*)^n \\
 &= V(h_{1,\gamma}(X, 0), \dots, h_{n,\gamma}(X, 0)) \cap (\mathbb{C}^*)^n = V_{0,\gamma}
 \end{aligned}$$

holds. Since $\det(\partial h_{i,\gamma}(X, 0)/\partial X_k)_{1 \leq i, k \leq n}(x_0^{(j,\gamma)}) \neq 0$ holds for $1 \leq j \leq D_\gamma$, we may apply of the global Newton iterator of [21] (see also [50]) in order to “lift” the given geometric solution of $V_{0,\gamma}$ to the geometric solution of the variety V_γ associated to the linear form $u \in \mathbb{Q}[X]$ with any prescribed precision.

Denote $m_{u,\gamma}^{(0)}, w_{u,1,\gamma}^{(0)}, \dots, w_{u,n,\gamma}^{(0)} \in \mathbb{Q}[Y]$ the polynomials which form the given geometric solution of $V_{0,\gamma}$, as provided by the algorithm underlying Theorem 4.3. Recall that $m_{u,\gamma}^{(0)}(u(x_0^{(j)})) = 0$ and $(x_0^{(j,\gamma)})_i = w_{u,i,\gamma}^{(0)}(u(x_0^{(j)}))$ holds for $1 \leq i \leq n$ and $1 \leq j \leq D_\gamma$. The global Newton iterator is a recursive procedure whose k th step computes approximations $m_{u,\gamma}^{(k)}, w_{u,1,\gamma}^{(k)}, \dots, w_{u,n,\gamma}^{(k)} \in \mathbb{Q}[T, Y]$ of the polynomials $m_{u,\gamma}, w_{u,1,\gamma}, \dots, w_{u,n,\gamma}$ which form the geometric solution of V_γ associated with the linear form u with precision 2^k for any $k \geq 0$.

Assume without loss of generality that $\gamma_i \geq 0$ and $0 = \min\{\gamma_1, \dots, \gamma_n\}$ hold for $1 \leq i \leq n$. Indeed, if there exists $\gamma_i < 0$, setting $\gamma_{i_0} := \min\{\gamma_1, \dots, \gamma_n\}$ we have

$$\begin{aligned} T^{-\gamma_{i_0} D_\gamma} m_{u,\gamma}(T^{\gamma_{n+1}}, T^{\gamma_{i_0}} Y) &= \prod_{j=1}^{D_\gamma} T^{-\gamma_{i_0}} \left(T^{\gamma_{i_0}} Y - \sum_{i=1}^n u_i \sum_{m \geq 0} x_{i,m}^{(j,\gamma)} T^{\gamma_i + m} \right) \\ &= \prod_{j=1}^{D_\gamma} \left(Y - T^{-\gamma_{i_0}} \sum_{i=1}^n u_i \sum_{m \geq 0} x_{i,m}^{(j,\gamma)} T^{\gamma_i + m} \right) \\ &= \prod_{j=1}^{D_\gamma} \left(Y - \sum_{i=1}^n u_i \sum_{m \geq 0} x_{i,m}^{(j,\gamma)} T^{\gamma_i - \gamma_{i_0} + m} \right). \end{aligned}$$

Since $\gamma_i - \gamma_{i_0} \geq 0$ holds for $1 \leq i \leq n$, this shows that the computation of an approximation $m_{u_\gamma} := m_\gamma(T^{\gamma_{n+1}}, Y)$ can be easily reduced to a situation in which $\gamma_i \geq 0$ holds for $1 \leq i \leq n$.

Note that the global Newton iterator cannot be directly applied in order to compute the geometric solution of $\{\sigma^{(j,\gamma)}; 1 \leq j \leq D_\gamma\}$ associated with the linear form $u_\gamma \in \mathbb{Q}[T][X]$, because the coefficients of u_γ are nonconstant polynomials of $\mathbb{Q}[T]$. Indeed, two critical problems arise:

- (1) Although by hypothesis u_γ separates the points of V_γ , it might not separate the points of $V_{0,\gamma}$ and it is not clear from which precision on, the corresponding approximations of the points of V_γ are separated by u_γ . Requiring u_γ to be a separating form for all the approximations of the points of V_γ is an essential hypothesis for the iterator of [21] which cannot be suppressed without causing a significant growth of the complexity of the procedure (see [30], [31]).
- (2) The iterator of [21] makes critical use of the fact that the coefficients of the linear form under consideration are elements of \mathbb{Q} in order to determine how a given precision can be achieved.

Nevertheless, we shall exhibit a modification of the procedure which computes an approximation of $m_{u_\gamma}(T, Y)$ with precision $2\gamma_{n+1}E$ without changing the asymptotic number of arithmetic operations performed.

In order to circumvent (1) we require an additional generic condition to be satisfied by the coefficients u_1, \dots, u_n defining $u_\gamma := \sum_{i=1}^n u_i T^{\gamma_i} X_i$. Recall that $u_\gamma(\sigma^{(j,\gamma)}) = \sum_{m \geq 0} \left(\sum_{i=1}^n u_i x_{i,m-\gamma_i}^{(j,\gamma)} \right) T^m$ for every $1 \leq j \leq D_\gamma$, where $x_{i,m-\gamma_i}^{(j,\gamma)} := 0$ for $m < \gamma_i$. To state this condition, we need the following claim:

Claim. Set $M_\gamma := \max\{\gamma_1, \dots, \gamma_n\}$ and let $\Lambda_1, \dots, \Lambda_n$ be indeterminates over $\mathbb{C}[T, X]$. Then the following inequality holds for every $1 \leq j, h \leq D_\gamma$ with $j \neq h$:

$$\sum_{m=0}^{M_\gamma} \left(\sum_{i=1}^n \Lambda_i x_{i,m-\gamma_i}^{(j,\gamma)} \right) T^m \neq \sum_{m=0}^{M_\gamma} \left(\sum_{i=1}^n \Lambda_i x_{i,m-\gamma_i}^{(h,\gamma)} \right) T^m.$$

Proof of Claim. Suppose on the contrary that there exist $j \neq h$ such that $\sum_{m=0}^{M_\gamma} \left(\sum_{i=1}^n \Lambda_i x_{i,m-\gamma_i}^{(j,\gamma)} \right) T^m = \sum_{m=0}^{M_\gamma} \left(\sum_{i=1}^n \Lambda_i x_{i,m-\gamma_i}^{(h,\gamma)} \right) T^m$. Substituting $T^{-\gamma_i} \Lambda_i$ for Λ_i in this identity for $i = 1, \dots, n$, we have $\sum_{m=0}^{M_\gamma} \sum_{i=1}^n \Lambda_i x_{i,m-\gamma_i}^{(j,\gamma)} T^{m-\gamma_i} = \sum_{m=0}^{M_\gamma} \sum_{i=1}^n \Lambda_i x_{i,m-\gamma_i}^{(h,\gamma)} T^{m-\gamma_i}$, that is

$$\sum_{i=1}^n \sum_{m=0}^{M_\gamma-\gamma_i} \Lambda_i x_{i,m}^{(j,\gamma)} T^m = \sum_{i=1}^n \sum_{m=0}^{M_\gamma-\gamma_i} \Lambda_i x_{i,m}^{(h,\gamma)} T^m.$$

Substituting 0 for T in this identity, we deduce that

$$\sum_{i=1}^n \Lambda_i x_{i,0}^{(j,\gamma)} = \sum_{i=1}^n \Lambda_i x_{i,0}^{(h,\gamma)},$$

which contradicts the fact that the vectors $\mathbf{x}_0^{(j,\gamma)} = (x_{1,0}^{(j,\gamma)}, \dots, x_{n,0}^{(j,\gamma)})$ ($1 \leq j \leq D_\gamma$) are all distinct. This finishes the proof of the claim.

By the claim we see that the polynomial $\sum_{m=0}^{M_\gamma} \left(\sum_{i=1}^n \Lambda_i (x_{i,m-\gamma_i}^{(j,\gamma)} - x_{i,m-\gamma_i}^{(h,\gamma)}) \right) T^m$ of $\mathbb{Q}[\Lambda][T]$ is nonzero, and therefore has a nonzero coefficient $a_{j,h} \in \mathbb{C}[\Lambda]$ for every $1 \leq j < h \leq D_\gamma$. Consider the polynomial $A_\gamma(\Lambda) := \prod_{1 \leq j < h \leq D_\gamma} a_{j,h} \in \mathbb{C}[\Lambda]$. Since $a_{j,h}$ has degree 1 for every $1 \leq j < h \leq D_\gamma$, it follows that A has degree $\binom{D_\gamma}{2}$. Furthermore, for every $(u_1, \dots, u_n) \in \mathbb{C}^n$ with $A_\gamma(u_1, \dots, u_n) \neq 0$, the corresponding polynomial $u_\gamma := \sum_{i=1}^n u_i T^{\gamma_i} X_i$ separates the initial terms $\sum_{m=0}^{M_\gamma} x_m^{(j,\gamma)} T^m$ of the power series $\sigma^{(j,\gamma)}$ ($1 \leq j \leq D_\gamma$).

From Theorem 2.2 we see that for a random choice of the coefficients u_1, \dots, u_n in the set $\{1, \dots, \rho D_\gamma^2\}$ the linear form u_γ separates the first M_γ terms of the points of V_γ with probability at least $1 - 1/\rho$. From now on we assume that u_γ satisfies this requirement.

The algorithm proceeds in three steps. First, it computes a suitable approximation to the geometric solution of V_γ associated to the linear form $u := \sum_{i=1}^n u_i X_i$ by means of $\kappa_0 := \lceil \log(M_\gamma + 1) \rceil$ steps of the global Newton iterator of [21]. This approximation is used in order to obtain the corresponding approximation $m_{u_\gamma}^{(\kappa_0)}, w_{u_\gamma,1}^{(\kappa_0)}, \dots, w_{u_\gamma,n}^{(\kappa_0)}$ of the polynomials that form the geometric solution of V_γ associated with u_γ . Finally, we apply an adaptation of the global Newton iterator which takes as input the polynomials of the previous step $m_{u_\gamma}^{(\kappa_0)}, w_{u_\gamma,1}^{(\kappa_0)}, \dots, w_{u_\gamma,n}^{(\kappa_0)}$ and outputs the required approximation to the polynomials $m_{u_\gamma}, w_{u_\gamma,1}, \dots, w_{u_\gamma,n}$ that form the geometric solution of V_γ associated with u_γ .

Now we consider the three steps above in detail. The first step takes as input the given geometric solution $m_{u,\gamma}^{(0)}, w_{u,1,\gamma}^{(0)}, \dots, w_{u,n,\gamma}^{(0)}$ of $V_{0,\gamma}$, and performs $\kappa_0 := \lceil \log(M_\gamma + 1) \rceil$ times the global Newton iterator of [21] to obtain polynomials $m_{u,\gamma}^{(\kappa_0)}, w_{u,1,\gamma}^{(\kappa_0)}, \dots, w_{u,n,\gamma}^{(\kappa_0)} \in \mathbb{Q}[T, Y]$ such that the following conditions hold:

- (i) $_{u,\kappa_0}$ $\deg_Y m_{u,\gamma}^{(\kappa_0)} = D_\gamma$ and $\deg_T m_{u,\gamma}^{(\kappa_0)} \leq M_\gamma$,
- (ii) $_{u,\kappa_0}$ $\deg_Y w_{u,i,\gamma}^{(\kappa_0)} < D_\gamma$ and $\deg_T w_{u,i,\gamma}^{(\kappa_0)} \leq M_\gamma$ for $1 \leq i \leq n$,

$$\begin{aligned} (iii)_{u, \kappa_0} \quad m_{u, \gamma}^{(\kappa_0)} &\equiv \prod_{j=1}^{D_\gamma} (Y - \varphi_{\kappa_0}^{(j, \gamma)}) \pmod{(T^{M_\gamma+1})}, \\ (iv)_{u, \kappa_0} \quad \sigma_i^{(j, \gamma)} &\equiv w_{u, i, \gamma}^{(\kappa_0)}(T, \varphi_{\kappa_0}^{(j, \gamma)}) \pmod{(T^{M_\gamma+1})} \text{ for } 1 \leq i \leq n. \end{aligned}$$

Here $\varphi_{\kappa_0}^{(j, \gamma)}$ is the Taylor expansion of order 2^{κ_0} of the power series $u(\sigma^{(j, \gamma)})$, that is, $\varphi_{\kappa_0}^{(j, \gamma)} := \sum_{m=0}^{2^{\kappa_0}} u(x_m^{(j, \gamma)}) T^m$ for $1 \leq j \leq D_\gamma$.

According to [21, Proposition 7], it follows that this step requires performing $O((nL_\gamma + n^\Omega)M(D_\gamma)M(M_\gamma))$ arithmetic operations in \mathbb{Q} , where L_γ denotes the number of arithmetic operations in \mathbb{Q} required to evaluate the polynomials $h_{i, \gamma}$ of (3.6). Furthermore, in view of the application of Lemma 2.4 it is important to remark that this step does not involve any division by a nonconstant polynomial in the coefficients u_1, \dots, u_n .

Next we discuss the second step. In this step we obtain approximations $m_{u_\gamma}^{(\kappa_0)}, w_{u_\gamma, 1}^{(\kappa_0)}, \dots, w_{u_\gamma, n}^{(\kappa_0)}$ of the polynomials that form the geometric solution of V_γ associated with u_γ with precision $2^{\kappa_0} \geq M_\gamma$, namely

- $\deg_Y m_{u_\gamma}^{(\kappa_0)} = D_\gamma$ and $\deg_T m_{u_\gamma}^{(\kappa_0)} \leq M_\gamma$,
- $\deg_Y w_{u_\gamma, i}^{(\kappa_0)} < D_\gamma$ and $\deg_T w_{u_\gamma, i}^{(\kappa_0)} \leq M_\gamma$ for $1 \leq i \leq n$,
- $m_{u_\gamma}^{(\kappa_0)} \equiv \prod_{j=1}^{D_\gamma} (Y - \phi_{\kappa_0}^{(j, \gamma)}) \pmod{(T^{M_\gamma+1})}$,
- $\sigma_i^{(j, \gamma)} \equiv w_{u_\gamma, i}^{(\kappa_0)}(T, \phi_{\kappa_0}^{(j, \gamma)}) \pmod{(T^{M_\gamma+1})}$ for $1 \leq i \leq n$.

Here $\phi_{\kappa_0}^{(j, \gamma)}$ is the Taylor expansion of $\phi^{(j, \gamma)} := u_\gamma(\sigma^{(j, \gamma)})$ of order 2^{κ_0} for $1 \leq j \leq D_\gamma$.

From conditions $(i)_{u, \kappa_0^-} (iv)_{u, \kappa_0}$ and the elementary properties of the resultant it is easy to see that $m_{u_\gamma}^{(\kappa_0)}$ satisfies the following identity:

$$(4.15) \quad m_{u_\gamma}^{(\kappa_0)}(Y) = \text{Res}_{\tilde{Y}} \left(Y - \sum_{i=1}^n u_i T^{\gamma_i} w_{u, i, \gamma}^{(\kappa_0)}(\tilde{Y}), m_{u_\gamma}^{(\kappa_0)}(\tilde{Y}) \right).$$

The resultant of the right-hand side is computed mod $(T^{M_\gamma+1})$ by interpolation in the variable Y to reduce the problem to the computation of D_γ resultants, as explained in the computation of the resultant in (4.7). These D_γ resultants involve two polynomials of $\mathbb{Q}[T, \tilde{Y}]$ of degree in \tilde{Y} bounded by D_γ and are computed mod $(T^{M_\gamma+1})$. Hence we deduce that this step requires $O(M(D_\gamma)D_\gamma M(M_\gamma))$ arithmetic operations in \mathbb{Q} .

We apply Lemma 2.4 in order to extend this procedure to an algorithm computing $m_{u_\gamma}^{(\kappa_0)}, w_{u_\gamma, 1}^{(\kappa_0)}, \dots, w_{u_\gamma, n}^{(\kappa_0)}$. For this purpose, we observe that a similar argument as in the proof of Proposition 4.1 proves that the denominators in $\mathbb{Q}[\Lambda]$ which arise during the computation of the D_γ resultants required to compute the minimal polynomial of the generic version $\sum_{i=1}^n \Lambda_i T^{\gamma_i} X_i$ of the linear form u_γ are divisors of a polynomial of $\mathbb{Q}[\Lambda]$ of degree at most $4D_\gamma^3$. Applying Theorem 2.2 we see that for a random choice of the coefficients u_1, \dots, u_n in the set $\{1, \dots, 4\rho D_\gamma^3\}$ none of these denominators are annihilated with probability at least $1 - 1/\rho$.

Finally, we consider the third step of the algorithm. For $\kappa_1 := \lceil \log(2\gamma_{n+1}E + 1) \rceil$, we apply $\kappa_1 - \kappa_0$ times an adaptation of the global Newton iterator of [21] to the polynomials $m_{u_\gamma}^{(\kappa_0)}, w_{u_\gamma, 1}^{(\kappa_0)}, \dots, w_{u_\gamma, n}^{(\kappa_0)}$ computed in the previous step. In the k th iteration step, we compute polynomials $m_{u_\gamma}^{(k)}, w_{u_\gamma, 1}^{(k)}, \dots, w_{u_\gamma, n}^{(k)}$ satisfying:

- $\deg_Y m_{u_\gamma}^{(k)} = D$ and $\deg_T m_{u_\gamma}^{(k)} \leq 2^k$,

- $m_{u_\gamma}^{(k)} = \prod_{j=1}^{D_\gamma} (Y - \phi_k^{(j,\gamma)}),$
- $\deg_Y w_{u_\gamma,i}^{(k)} < D$ and $\deg_T w_{u_\gamma,1}^{(k)} \leq 2^k$ for $1 \leq i \leq n,$
- $\sigma_i^{(j,\gamma)} \equiv w_{u_\gamma,i}^{(k)}(T, \phi_k^{(j,\gamma)}) \pmod{(T^{2^k+1})}$ for $1 \leq i \leq n.$

Here $\phi_k^{(j,\gamma)}$ is the Taylor expansion of $\phi^{(j,\gamma)} := u_\gamma(\sigma^{(j,\gamma)})$ of order 2^k for $1 \leq j \leq D_\gamma$. In particular, it follows that $m_{u_\gamma}^{(\kappa_1)}$ is the required approximation to m_{u_γ} with precision $2\gamma_{n+1}E$.

Fix $\kappa_0 < k \leq \kappa_1$. We briefly describe how we can obtain an approximation with precision 2^k of the polynomials that form the geometric solution of V_γ associated to the linear form u_γ from an approximation with precision 2^{k-1} . Similarly to [21], set $\Delta_k(T, Y) := u_\gamma(\tilde{w}_{u_\gamma}^{(k)}) - u_\gamma(w_{u_\gamma}^{(k-1)}) = u_\gamma(\tilde{w}_{u_\gamma}^{(k)}) - Y$, where $\tilde{w}_{u_\gamma}^{(k)}$ is the result of applying a “classical Newton step” to $w_{u_\gamma}^{(k-1)}$, as described in [21]. Furthermore, write $\Delta_m(T, Y) := T^{-1-2^{k-1}}(m_{u_\gamma}^{(k)} - m_{u_\gamma}^{(k-1)})$. Since $m_{u_\gamma}^{(k)}(Y + \Delta_k) \equiv 0 \pmod{(T^{2^k+1}, m_{u_\gamma}^{(k-1)})}$ holds (see [15, §4.2]), it follows that

$$\begin{aligned} 0 &\equiv m_{u_\gamma}^{(k)}(Y + \Delta_k) \equiv m_{u_\gamma}^{(k-1)}(Y + \Delta_k) + T^{2^{k-1}+1} \Delta_m(Y + \Delta_k) \pmod{(T^{2^k+1}, m_{u_\gamma}^{(k-1)})} \\ &\equiv \Delta_k \frac{\partial m_{u_\gamma}^{(k-1)}}{\partial Y}(Y) + T^{2^{k-1}+1} \Delta_m(Y) \pmod{(T^{2^k+1}, m_{u_\gamma}^{(k-1)})}. \end{aligned}$$

We conclude that the following congruence relation holds:

$$(4.16) \quad m_{u_\gamma}^{(k)} \equiv m_{u_\gamma}^{(k-1)} - \left(\Delta_k \frac{\partial m_{u_\gamma}^{(k-1)}}{\partial Y} \pmod{m_{u_\gamma}^{(k-1)}} \right) \pmod{(T^{2^k+1})}.$$

A similar argument proves the following congruence relation

$$(4.17) \quad w_{u_\gamma,i}^{(k)} \equiv \tilde{w}_{u_\gamma,i}^{(k-1)} - \left(\Delta_k \frac{\partial \tilde{w}_{u_\gamma,i}^{(k-1)}}{\partial Y} \pmod{m_{u_\gamma}^{(k-1)}} \right) \pmod{(T^{2^k+1})} \text{ for } 1 \leq i \leq n.$$

Each iteration of our adaptation of the global Newton iteration is based on (4.16) and (4.17), which are extensions of the corresponding congruence relations of [21]. We first compute $\tilde{w}_{u_\gamma}^{(k)}$ by a standard Newton–Hensel lifting, and then evaluate the expressions (4.16) and (4.17). With a similar analysis as in [21, Proposition 7] we conclude that the whole procedure requires $O((nL_\gamma + n^\Omega)\mathbf{M}(D_\gamma)E\gamma_{n+1})$ arithmetic operations in \mathbb{Q} .

Finally, combining the complexity estimates of the three steps above and the probability of achievement of the two generic conditions imposed to the coefficients u_1, \dots, u_n , we deduce the statement of the proposition. \square

Using the algorithm of the statement of Proposition 4.6 for all $\gamma \in \Gamma$ we obtain approximations of the factors m_γ which allow us to compute the minimal polynomial m_u and hence a geometric solution of \hat{V} . Our next result outlines this procedure and estimates its complexity and error probability.

Proposition 4.7. *Suppose that we are given a geometric solution of the variety $V_{0,\gamma}$ for all $\gamma \in \Gamma$, as provided by Theorem 4.3, with a linear form $u \in \mathbb{Q}[X_1, \dots, X_n]$ whose coefficients are randomly chosen in the set $\{1, \dots, 4\rho D^4\}$, where ρ is a fixed positive integer. Then there is an algorithm which computes a geometric solution of the curve \hat{V} with error probability bounded by $1/\rho$ performing $O((n^2L + n^{1+\Omega})\mathbf{M}(\mathcal{M}_\Gamma)\mathbf{M}(D)(\mathbf{M}(D) + \mathbf{M}(E)))$ arithmetic operations in \mathbb{Q} . Here*

$L := \max_{\gamma \in \Gamma} L_\gamma$, where L_γ is the number of arithmetic operations required to evaluate the polynomials $h_{i,\gamma}$ of (3.6) for all $\gamma \in \Gamma$ and $\mathcal{M}_\Gamma := \max_{\gamma \in \Gamma} \max\{\gamma_1, \dots, \gamma_{n+1}\}$.

Proof. For each $\gamma \in \Gamma$, we apply the algorithm underlying the proof of Proposition 4.6 in order to obtain an approximation of m_{u_γ} with precision $2\gamma_{n+1}E$. Due to (4.14), this polynomial immediately yields an approximation with precision $2E$ of $m_\gamma(T, Y)$ in $\mathbb{Q}((T^{1/\gamma_{n+1}}))[Y]$.

Multiplying all these approximations, we obtain an approximation with precision $2E$ of the polynomial $\hat{m}_u = \prod_{\gamma \in \Gamma} m_\gamma$ of (4.12). Since every coefficient $a_j(T)$ of $\hat{m}_u \in \mathbb{Q}(T)[Y]$ is a rational function of $\mathbb{Q}(T)$ having a reduced representation with numerator and denominator of degree at most E , such a representation of $a_j(T)$ can be computed from its approximation with precision $2E$ using Padé approximation with $O(M(E))$ arithmetic operations in \mathbb{Q} .

In order to estimate the complexity of the whole procedure, we estimate the complexity of its three main steps:

- (i) the computation of the polynomials m_γ with precision $2E$ for all $\gamma \in \Gamma$, which requires $O(\sum_{\gamma \in \Gamma} (nL_\gamma + n^\Omega)M(D_\gamma)(M(M_\gamma)M(D_\gamma) + E\gamma_{n+1}))$ arithmetic operations in \mathbb{Q} ,
- (ii) the computation of the product $\prod_{\gamma \in \Gamma} m_\gamma$ with precision $2E$, which requires $O(M(D)M(E))$ arithmetic operations in \mathbb{Q} ,
- (iii) the computation of a reduced representation of all the coefficients of $\hat{m}_u \in \mathbb{Q}(T)[Y]$, which requires $O(M(E)D)$ arithmetic operations in \mathbb{Q} .

In conclusion, the algorithm performs $O((nL + n^\Omega)M(\mathcal{M}_\Gamma)M(D)(M(D) + M(E)))$ arithmetic operations in \mathbb{Q} , where $\mathcal{M}_\Gamma := \max_{\gamma \in \Gamma} \{M_\gamma, \gamma_{n+1}\}$ and $L := \max_{\gamma \in \Gamma} L_\gamma$.

Next we discuss how this procedure can be extended to the computation of a geometric solution of \hat{V} in the sense of Section 2.3. Two computations of the above procedure involve divisions by the coefficients u_i of the linear form u : the computation of the resultant of (4.15) for all $\gamma \in \Gamma$ and the Padé approximations of (iii). Both computations are reduced to D applications of the EEA, which is performed in a ring $\mathbb{Q}(\Lambda)$. A similar analysis as in Proposition 4.1 shows that all the denominators in $\mathbb{Q}[\Lambda]$ arising during such application of the EEA are divisors of a polynomial of degree $4D^4$. Therefore, according to Lemma 2.4, we conclude that a geometric solution of \hat{V} can be computed with $O((n^2L + n^{1+\Omega})M(\mathcal{M}_\Gamma)M(D)(M(D) + M(E)))$ arithmetic operations in \mathbb{Q} , with an algorithm with error probability at most $1/\rho$, provided that the coefficients of u are randomly chosen in the set $\{1, \dots, 4\rho D^4\}$. \square

Putting together Theorem 4.3 and Proposition 4.7 we obtain the main result of this section:

Theorem 4.8. *Let ρ be a fixed positive integer. Suppose that the coefficients of the linear form \hat{u} of the statement of Theorem 4.3 and of the linear form u are randomly chosen in the set $\{1, \dots, 4\rho D^4\}$. Then the algorithm underlying Theorem 4.3 and Proposition 4.7 computes a geometric solution of the curve \hat{V} with error probability $3/\rho$ performing $O((n^2L + n^{1+\Omega})M(\mathcal{M}_\Gamma)\log(\mathcal{Q})M(D)(M(D) + M(E)))$ arithmetic operations in \mathbb{Q} . Here $L := \max_{\gamma \in \Gamma} L_\gamma$, where L_γ is the number of arithmetic operations required to evaluate the polynomials $h_{i,\gamma}$ of (3.6) for all $\gamma \in \Gamma$, $\mathcal{Q} := 2 \max_{1 \leq i \leq n} \{\|q\|; q \in \Delta_i\}$, and $\mathcal{M}_\Gamma := \max_{\gamma \in \Gamma} \|\gamma\|$.*

4.3. Solving a sufficiently generic sparse system. Now we obtain a geometric solution of the zero-dimensional variety $V_1 := \{x \in \mathbb{C}^n : h_1(x) = 0, \dots, h_n(x) = 0\}$ from a geometric solution of the curve \widehat{V} .

With notations as in the previous section, we have that $V_1 = \pi^{-1}(1)$, where $\pi : \widehat{V} \rightarrow \mathbb{A}^1$ is the linear projection defined by $\pi(x, t) := t$. Moreover, due to Lemma 4.5, the equality $V_1 = \pi^{-1}(1) \cap \widehat{V}$ holds.

This enables us to easily obtain a geometric solution of V_1 from a geometric solution of the curve \widehat{V} . Indeed, let $\widehat{m}_u(T, Y), \widehat{v}_1(T, Y), \dots, \widehat{v}_n(T, Y)$ be the polynomials which form a geometric solution of \widehat{V} associated to a linear form $u \in \mathbb{Q}[X]$. Suppose further that the linear form u separates the points of V_1 . Making the substitution $T = 1$, we obtain new polynomials $\widehat{m}_u(1, Y), \widehat{v}_1(1, Y), \dots, \widehat{v}_n(1, Y) \in \mathbb{Q}[Y]$ such that $\widehat{m}_u(1, u(X))$ and $\frac{\partial \widehat{m}_u}{\partial Y}(1, u(X))X_i - \widehat{v}_i(1, u(X))$ ($1 \leq i \leq n$) vanish over V_1 . Taking into account that $\deg_Y(m_u) = D = \#V_1$ and that u separates the points of V_1 , it follows that the polynomials $\widehat{m}_u(1, Y), \widehat{v}_1(1, Y), \dots, \widehat{v}_n(1, Y) \in \mathbb{Q}[Y]$ form a geometric solution of V_1 .

Proposition 4.9. *Let ρ be a fixed positive integer. With assumptions and notations as in Theorem 4.8, the algorithm described above computes a geometric solution of the zero-dimensional variety V_1 with error probability $4/\rho$ using $O((n^2L + n^{1+\Omega})M(\mathcal{M}_\Gamma) \log(\mathcal{Q})M(D)(M(D) + M(E)))$ arithmetic operations in \mathbb{Q} .*

5. THE SOLUTION OF THE ORIGINAL SYSTEM

Let notations and assumptions be as in the previous sections. Assume that we are given a geometric solution $m_u(Y), v_1(Y), \dots, v_n(Y)$ of the zero-dimensional variety V_1 defined by the polynomials $h_1 := f_1 + g_1, \dots, h_n := f_n + g_n$. Assume further that the linear form u of such a geometric solution separates the points of the zero-dimensional variety $f_1 = \dots = f_n = 0$. In this section we describe a procedure for computing a geometric solution of the input system $f_1 = \dots = f_n = 0$.

For this purpose, we introduce an indeterminate T over $\mathbb{Q}[X]$ and consider the “deformation” $F_1, \dots, F_n \in \mathbb{Q}[X, T]$ of the polynomials f_1, \dots, f_n defined in the following way:

$$F_i(X, T) := f_i(X) + (1 - T)g_i(X) \quad (1 \leq i \leq n).$$

Set $\mathcal{V} := \{(x, t) \in \mathbb{A}^{n+1} : F_1(x, t) = \dots = F_n(x, t) = 0\}$ and denote by $\pi : \mathcal{V} \rightarrow \mathbb{A}^1$ the projection map defined by $\pi(x, t) := t$. As in Subsection 4.2, we introduce the variety $\mathcal{V}_{\text{dom}} \subset \mathbb{A}^{n+1}$ defined as the union of all the irreducible components of \mathcal{V} whose projection over \mathbb{A}^1 is dominant.

5.1. Solution of the second deformation. In this section we describe an efficient procedure for computing a geometric solution of \mathcal{V}_{dom} from the geometric solution of $\pi^{-1}(0)$ provided by Proposition 4.9.

Since $\pi^{-1}(0)$ is the variety defined by the “sufficiently generic” sparse system $h_1(X) = F_1(X, 0) = 0, \dots, h_n(X) = F_n(X, 0) = 0$, with similar arguments to those leading to the proof of Lemma 4.5, it is not difficult to see that the polynomials F_1, \dots, F_n , the variety \mathcal{V} , the projection $\pi : \mathcal{V} \rightarrow \mathbb{A}^1$, and the fiber $\pi^{-1}(0)$ satisfy all the assumptions of Lemma 4.4. We conclude that \mathcal{V}_{dom} is a curve and that the identity $\mathcal{V} \cap \pi^{-1}(0) = \mathcal{V}_{\text{dom}} \cap \pi^{-1}(0)$ holds. Furthermore, Lemma 4.4 implies that all the hypotheses of [50, Theorem 2] are satisfied.

Therefore, applying the “formal Newton lifting process” underlying the proof of [50, Theorem 2], we compute polynomials $m(T, Y), v_1(T, Y), \dots, v_n(T, Y) \in \mathbb{Q}[T, Y]$ which form a geometric solution of \mathcal{V}_{dom} . The formal Newton lifting process requires $O((nL' + n^{\Omega+1})M(D)M(E'))$ arithmetic operations in \mathbb{Q} , where L' denotes the number of arithmetic operations required to evaluate F_1, \dots, F_n and E' is any upper bound of the degree of m in the variable T .

We can apply Lemma 2.3 in order to estimate $\deg_T m$ in combinatorial terms. Indeed, let $\tilde{Q}_1, \dots, \tilde{Q}_n \subset \mathbb{R}^{n+1}$ be the Newton polytopes of the polynomials F_1, \dots, F_n and let $\Delta \subset \mathbb{R}^{n+1}$ be the standard unitary simplex in the plane $\{T = 0\}$. Since $\tilde{Q}_i \subset Q_i \times [0, 1]$ holds for $1 \leq i \leq n$, where $Q_i \subset \mathbb{R}^n$ is the Newton polytope of h_i , by (2.3) of Lemma 2.3 we deduce the following estimate:

$$(5.1) \quad \deg_T m_u \leq E' := \sum_{i=1}^n MV(\Delta, Q_1, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_n).$$

With this definition of E' , we have:

Proposition 5.1. *Suppose that we are given a geometric solution of the variety V_1 , as provided by Proposition 4.9. A geometric solution of \mathcal{V}_{dom} can be deterministically computed with $O((nL' + n^{\Omega+1})M(D)M(E'))$ arithmetic operations in \mathbb{Q} .*

5.2. Solving the input system. Making the substitution $T = 1$ in the polynomials $m(T, Y), v_i(T, Y)$ ($1 \leq i \leq n$) which form the geometric solution of \mathcal{V}_{dom} computed by the algorithm of Proposition 5.1 we obtain polynomials $m(1, Y), v_1(1, Y), \dots, v_n(1, Y) \in \mathbb{Q}[Y]$ which represent a complete description of our input system $f_1(X) = \dots = f_n(X) = 0$, eventually including multiplicities. Such multiplicities are represented by multiple factors of $m(1, Y)$, which are also factors of $v_1(1, Y), \dots, v_n(1, Y)$ (see e.g. [21, §6.5]). In order to remove them, we compute $a(Y) := \gcd(m(1, Y), (\partial m / \partial Y)(1, Y))$, and the polynomials $m(1, Y)/a(Y)$, $(\partial m / \partial Y)(1, Y)/a(Y)$, $v_i(1, Y)/a(Y)$ ($1 \leq i \leq n$). These polynomials form a geometric solution of our input system and can be computed with $O(nM(D)E')$ additional arithmetic operations in \mathbb{Q} .

Summarizing, we sketch the whole procedure computing a geometric solution of the input system $f_1 = \dots = f_n = 0$. Fix $\rho \geq 4$. We randomly choose the coefficients of the polynomials g_1, \dots, g_n in the set $\{1, \dots, 4\rho(nd)^{2n+1} + 2\rho n^2 2^{\mathcal{N}_1 + \dots + \mathcal{N}_s}\}$ and coefficients of linear forms u, \tilde{u} in the set $\{1, \dots, 16n\rho D^4\}$. By Theorem 2.2 it follows that the polynomials g_1, \dots, g_n and the linear forms u, \tilde{u} satisfy all the conditions required with probability at least $1 - 1/\rho$. Then we apply the algorithms underlying Propositions 4.9 and 5.1 in order to obtain a geometric solution of the variety \mathcal{V}_{dom} . Finally, we use the procedure above to compute a geometric solution of the input system $f_1 = \dots = f_n = 0$. This yields the following result:

Theorem 5.2. *The algorithm sketched above computes a geometric solution of the input system $f_1 = \dots = f_n = 0$ with error probability at most $1/\rho$ using*

$$O\left((n^2 \max\{L, L'\} + n^{1+\Omega}) M(D) (\log(\mathcal{Q})M(\mathcal{M}_\Gamma)(M(D) + M(E)) + M(E'))\right)$$

arithmetic operations in \mathbb{Q} . Here $L := \max_{\gamma \in \Gamma} L_\gamma$, where L_γ is the number of arithmetic operations required to evaluate the polynomials $h_{i,\gamma}$ of (3.6) for all $\gamma \in \Gamma$, L' denotes the number of arithmetic operations required to evaluate F_1, \dots, F_n , $\mathcal{M}_\Gamma := \max_{\gamma \in \Gamma} \|\gamma\|$, and $\mathcal{Q} := 2 \max_{1 \leq i \leq n} \{\|q\|; q \in \Delta_i\}$.

REFERENCES

- [1] E.L. Allgower and K. Georg, *Numerical continuation methods: An introduction*, Springer Ser. Comput. Math., vol. 13, Springer, New York, 1990.
- [2] M.E. Alonso, E. Becker, M.-F. Roy, and T. Wörmann, *Zeros, multiplicities and idempotents for zerodimensional systems*, Algorithms in Algebraic Geometry and Applications, Proceedings of MEGA'94 (Boston), Progr. Math., vol. 143, Birkhäuser Boston, 1996, pp. 1–15.
- [3] D. Bayer and D. Mumford, *What can be computed in algebraic geometry?*, Computational Algebraic Geometry and Commutative Algebra (Cambridge) (D. Eisenbud and L. Robbiano, eds.), Sympos. Math., vol. XXXIV, Cambridge Univ. Press, 1993, pp. 1–49.
- [4] D.N. Bernstein, *The number of roots of a system of equations*, Funct. Anal. Appl. **9** (1975), 183–185.
- [5] D. Bini and V. Pan, *Polynomial and matrix computations*, Progress in Theoretical Computer Science, Birkhäuser, Boston, 1994.
- [6] L. Blum, F. Cucker, M. Shub, and S. Smale, *Complexity and real computation*, Springer, New York Berlin Heidelberg, 1998.
- [7] A. Bompadre, G. Matera, R. Wachenchauer, and A. Waissbein, *Polynomial equation solving by lifting procedures for ramified fibers*, Theoret. Comput. Sci. **315** (2004), no. 2–3, 335–369.
- [8] A. Borodin, *Time space tradeoffs (getting closer to the barriers?)*, 4th International Symposium on Algorithms and Computation, ISAAC '93, Hong Kong, December 15–17, 1993 (Berlin), Lecture Notes in Comput. Sci., vol. 762, Springer, 1993, pp. 209–220.
- [9] A. Bostan, G. Lecerf, and E. Schost, *Tellegen's principle into practice*, Proceedings of the International Symposium on Symbolic and Algebraic Computation (ISSAC'03) (Philadelphia, USA, August 3–6, 2003) (New York) (J.R. Sendra, ed.), ACM Press, 2003, pp. 37–44.
- [10] A. Bostan and E. Schost, *Polynomial evaluation and interpolation on special sets of points*, J. Complexity **21** (2005), no. 4, 420–446.
- [11] P. Bürgisser, M. Clausen, and M.A. Shokrollahi, *Algebraic complexity theory*, Grundlehren Math. Wiss., vol. 315, Springer, Berlin, 1997.
- [12] A. Cafure, G. Matera, and A. Waissbein, *Inverting bijective polynomial maps over finite fields*, Proceedings of the 2006 Information Theory Workshop, ITW2006 (Punta del Este, Uruguay, March 13–17, 2006) (G. Seroussi and A. Viola, eds.), IEEE Information Theory Society, 2006, pp. 27–31.
- [13] D. Castro, M. Giusti, J. Heintz, G. Matera, and L.M. Pardo, *The hardness of polynomial equation solving*, Found. Comput. Math. **3** (2003), no. 4, 347–420.
- [14] D. Cox, J. Little, and D. O'Shea, *Using algebraic geometry*, Grad. Texts in Math., vol. 185, Springer, New York, 1998.
- [15] C. Durvy and G. Lecerf, *A concise proof of the Kronecker polynomial system solver from scratch*, Manuscript Université de Versailles Saint-Quentin-en-Yvelines, France, 2006.
- [16] I.Z. Emiris and J. Canny, *Efficient incremental algorithms for the sparse resultant and the mixed volume*, J. Symbolic Comput. **20** (1995), 117–149.
- [17] G. Ewald, *Combinatorial convexity and algebraic geometry*, Grad. Texts in Math., vol. 168, Springer, New York, 1996.
- [18] I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Birkhäuser, Boston, 1994.
- [19] M. Giusti, K. Hägele, J. Heintz, J.E. Morais, J.L. Montaña, and L.M. Pardo, *Lower bounds for Diophantine approximation*, J. Pure Appl. Algebra **117,118** (1997), 277–317.
- [20] M. Giusti, J. Heintz, J.E. Morais, J. Morgenstern, and L.M. Pardo, *Straight-line programs in geometric elimination theory*, J. Pure Appl. Algebra **124** (1998), 101–146.
- [21] M. Giusti, G. Lecerf, and B. Salvy, *A Gröbner free alternative for polynomial system solving*, J. Complexity **17** (2001), no. 1, 154–211.
- [22] J. Heintz, *On the computational complexity of polynomials and bilinear mappings. A survey*, Proceedings 5th International Symposium on Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, AAEECC-5, Menorca, Spain, June 15–19, 1987 (Berlin) (L. Huguet and A. Poli, eds.), Lecture Notes in Comput. Sci., vol. 356, Springer, 1989, pp. 269–300.
- [23] J. Heintz, G. Jeronimo, J. Sabia, J. San Martín, and P. Solernó, *Intersection theory and deformation algorithms. The multihomogeneous case*, Manuscript Universidad de Buenos Aires, 2002.

- [24] J. Heintz, T. Krick, S. Puddu, J. Sabia, and A. Waissbein, *Deformation techniques for efficient polynomial equation solving*, J. Complexity **16** (2000), no. 1, 70–109.
- [25] B. Huber and B. Sturmfels, *A polyhedral method for solving sparse polynomial systems*, Math. Comp. **64** (1995), no. 112, 1541–1555.
- [26] ———, *Bernstein’s Theorem in affine space*, Discrete Comput. Geom. **17** (1997), 137–141.
- [27] G. Jeronimo, T. Krick, J. Sabia, and M. Sombra, *The computational complexity of the Chow form*, Found. Comput. Math. **4** (2004), no. 1, 41–117.
- [28] A.G. Khovanski, *Newton polyhedra and the genus of complete intersections*, Funct. Anal. Appl. **12** (1978), 38–46.
- [29] A.G. Kushnirenko, *Newton polytopes and the Bézout Theorem*, Funct. Anal. Appl. **10** (1976), 233–235.
- [30] G. Lecerf, *Quadratic Newton iteration for systems with multiplicity*, Found. Comput. Math. **2** (2002), no. 3, 247–293.
- [31] ———, *Computing the equidimensional decomposition of an algebraic closed set by means of lifting fibers*, J. Complexity **19** (2003), no. 4, 564–596.
- [32] T.Y. Li, *Numerical solution of multivariate polynomial systems by homotopy continuation methods*, Acta Numer. **6** (1997), 399–436.
- [33] T.Y. Li and X. Wang, *The BKK root count in \mathbb{C}^n* , Math. Comp. **65** (1996), no. 216, 1477–1484.
- [34] R. Lidl and H. Niederreiter, *Finite fields*, Addison–Wesley, Reading, Massachusetts, 1983.
- [35] A. Morgan, *Solving polynomial systems using continuation for engineering and scientific problems*, Prentice–Hall, Englewood Cliffs, N.J., 1987.
- [36] A. Morgan, A. Sommese, and C. Wampler, *A generic product–decomposition formula for Bézout numbers*, SIAM J. Numer. Anal. **32** (1995), 1308–1325.
- [37] M. Oka, *Non-degenerate complete intersection singularity*, Hermann, Paris, 1997.
- [38] L.M. Pardo, *How lower and upper complexity bounds meet in elimination theory*, Applied Algebra, Algebraic Algorithms and Error Correcting Codes, Proceedings of AAECC–11 (Berlin) (G. Cohen, M. Giusti, and T. Mora, eds.), Lecture Notes in Comput. Sci., vol. 948, Springer, 1995, pp. 33–69.
- [39] L.M. Pardo and J. San Martín, *Deformation techniques to solve generalized Pham systems*, Theoret. Comput. Sci. **315** (2004), no. 2–3, 593–625.
- [40] P. Pedersen and B. Sturmfels, *Product formulas for resultants and Chow forms*, Math. Z. **214** (1993), no. 3, 377–396.
- [41] P. Philippon and M. Sombra, *Hauteur normalisée des variétés toriques projectives*, eprint math.NT/0406476, 38pp., 2003.
- [42] ———, *Géométrie diophantienne et variétés toriques*, C. R. Math. Acad. Sci. Paris **340** (2005), 507–512.
- [43] ———, *Mixed integrals and an analog of Bernstein theorem in dimension 1*, Manuscript, 2006.
- [44] J.M. Rojas, *Solving degenerate sparse polynomial systems faster*, J. Symbolic Comput. **28** (1999), no. 1/2, 155–186.
- [45] ———, *Algebraic geometry over four rings and the frontier of tractability*, Proceedings of a Conference on Hilbert’s Tenth Problem and Related Subjects (University of Gent, November 1–5, 1999) (Providence, RI) (J. Denef et al., ed.), Contemp. Math., vol. 270, Amer. Math. Soc., 2000, pp. 275–321.
- [46] M. Rojas and X. Wang, *Counting affine roots of polynomial systems via pointed Newton polytopes*, J. Complexity **12** (1996), no. 2, 116–133.
- [47] J. Sabia and P. Solernó, *Bounds for traces in complete intersections and degrees in the Nullstellensatz*, Appl. Algebra Engrg. Comm. Comput. **6** (1996), no. 6, 353–376.
- [48] J.E. Savage, *Models of computation. Exploring the power of computing*, Addison Wesley, Reading, Massachusetts, 1998.
- [49] W. Schmidt, *Equations over finite fields. An elementary approach*, Lectures Notes in Math., no. 536, Springer, New York, 1976.
- [50] E. Schost, *Computing parametric geometric resolutions*, Appl. Algebra Engrg. Comm. Comput. **13** (2003), 349–393.
- [51] A. Storjohann, *Algorithms for matrix canonical forms*, Ph.D. thesis, ETH, Zürich, Switzerland, 2000.

- [52] V. Strassen, *Algebraic complexity theory*, Handbook of Theoretical Computer Science (J. van Leeuwen, ed.), Elsevier, Amsterdam, 1990, pp. 634–671.
- [53] J. Verschelde, K. Gatermann, and R. Cools, *Mixed volume computation by dynamic lifting applied to polynomial system solving*, Discrete Comput. Geom. **16** (1996), no. 1, 69–112.
- [54] J. Verschelde, P. Verlinden, and R. Cools, *Homotopies exploiting Newton polytopes for solving sparse polynomial systems*, SIAM J. Numer. Anal. **31** (1994), no. 3, 915–930.
- [55] J. von zur Gathen, *Parallel arithmetic computations: a survey*, Proceedings of the 12th International Symposium on Mathematical Foundations of Computer Science, Bratislava, Czechoslovakia, August 25–29, 1996 (Berlin) (J. Gruska, B. Rován, and J. Wiedermann, eds.), Lecture Notes in Comput. Sci., vol. 233, Springer, August 1986, pp. 93–112.
- [56] J. von zur Gathen and J. Gerhard, *Modern computer algebra*, Cambridge Univ. Press, Cambridge, 1999.
- [57] R.J. Walker, *Algebraic curves*, Dover Publications Inc., New York, 1950.

¹DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS Y NATURALES, UNIVERSIDAD DE BUENOS AIRES, CIUDAD UNIVERSITARIA, PABELLÓN I (1428) BUENOS AIRES, ARGENTINA.
E-mail address: `jeronimo@dm.uba.ar`, `psolerno@dm.uba.ar`

² NATIONAL COUNCIL OF SCIENCE AND TECHNOLOGY (CONICET), ARGENTINA.

³INSTITUTO DE DESARROLLO HUMANO, UNIVERSIDAD NACIONAL DE GENERAL SARMIENTO, J.M. GUTIÉRREZ 1150 (1613) LOS POLVORINES, BUENOS AIRES, ARGENTINA.
E-mail address: `gmatera@ungs.edu.ar`

⁴CORELABS, CORE ST, HUMBOLDT 1967 (C1414CTU) CIUDAD DE BUENOS AIRES, ARGENTINA.

⁵DOCTORADO EN INGENIERÍA, INSTITUTO TECNOLÓGICO DE BUENOS AIRES, AV. EDUARDO MADERO 399 (C1106ACD) CIUDAD DE BUENOS AIRES, ARGENTINA.
E-mail address: `Ariel.Waissbein@corest.com`